Linear Control Systems (20ECE241)

Unit-I

Modeling of Control Systems.

Introduction:

Automatic control of many day-to-day tasks relieves the human beings from performing repetitive manual operation. Automatic control allows optimal performance of dynamic systems, increases productivity enormously, removes drudgery of performing same task again and again.

The modern engineers and scientists must, therefore, have a thorough knowledge of the principles of automatic control systems.

Control System and types:

The basic principles developed in control theory were not only used in engineering applications, but also in non-engineering systems like economic, socio economic systems and biological systems.

Types:

1. Open loop control System
2. Closed loop control system.

1. Open loop control systems:

A system in which the output has no effect on the control action is known as an open loop control system.

For a given input, the system produces a certain output.

If there is any disturbance, the output Changes, which cannot be adjusted to bring back the original output.

eg: A traffic control system.
A washing machine. etc.

Advantages:

1. They are simple and easy to build.
2. Cheaper, as they use less number of Components to build
3. Usually stable
4. Maintenance is easy.

Disadvantages:

1. They are less accurate
2. If external disturbances present, output differs significantly from the desired value

2. Closed-loop Control System:

A system in which the output is measured and compared with the reference input and an error signal is generated accordingly the output is controlled is called a closed-loop control system.

This is also known as feedback control system.

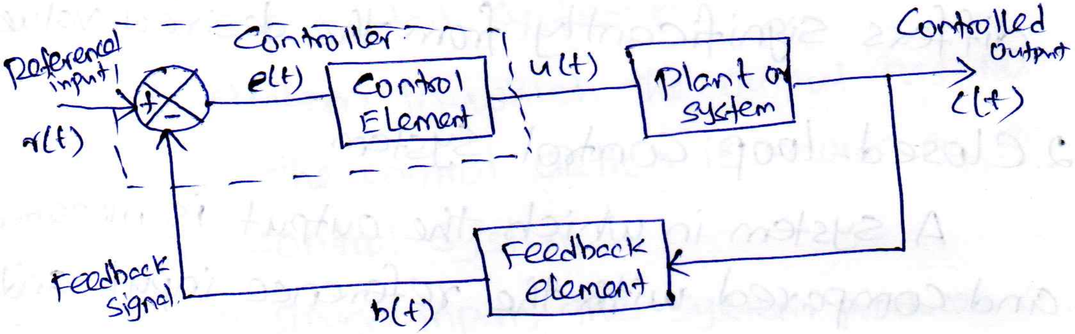
Advantages:

1. More accurate system.
2. effect of external disturbances can be made very small.
3. speed of the response can be greatly increased.

Disadvantages:

1. More complex and expensive

2. The system is prone to instability. Oscillations in the output may occur.
3. High maintenance cost.



Effects of feed back:

The input to the entire system is called as a reference input or a command input, r(t).

An error detector senses the difference between the reference input and the feedback signal equal to or proportional to the controlled output.

The feedback elements measure the controlled output and convert or transform it to a suitable value so that it can be compared with the reference input.

If the feedback signal, b(t) is equal to the Controlled output, c(t), the feedback system is called as unity feedback system.

The difference between the reference input and the feedback signal is known as the error signal or actuating signal e(t).

This signal manipulates the system or plant dynamics so that the desired output is obtained.

The controller acts until the error between the output variable and the reference input is zero.

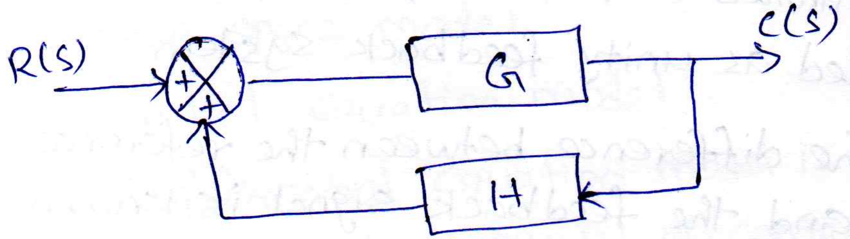
There are two types of feedback ~~Differential equation model.~~

i) Positive feedback

ii) Negative feedback

i) Positive feed back:

The positive feedback adds the reference input, and the feedback output. The following figure shows the block diagram of positive feedback control system.



The transfer function of positive feedback control system is,

$$T = \frac{G}{1-GH} \quad \text{—— ①}$$

Where,

T → the transfer function or overall gain of positive feedback system.

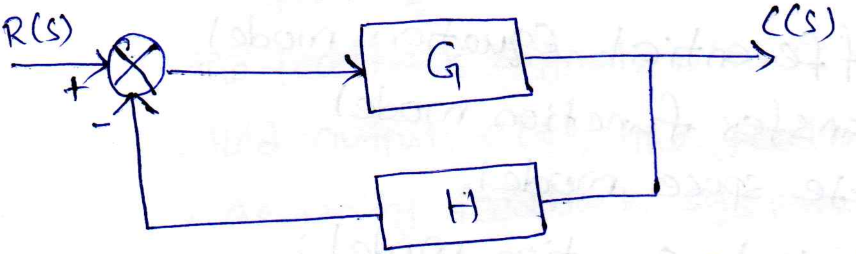
G → open-loop gain

H → gain of feedback path.

ii) Negative feedback:

Negative feedback reduces the error between the reference input and system output.

The following figure shows the block diagram of the negative feedback control system.



The transfer function of negative feedback system is,

$$T = \frac{G}{1+GH} \quad ②$$

where,

T → Transfer function or ~~[illegible]~~ overall gain of negative feedback system.

G → open-loop gain

H → gain of feedback path.

Differential equation model:

The control systems can be represented with a set of mathematical equations known as mathematical model.

These models are useful for analysis and design of control systems.

The following mathematical models are mostly used:

- Differential equation model
- Transfer function model
- State space model

i) Differential equation model:

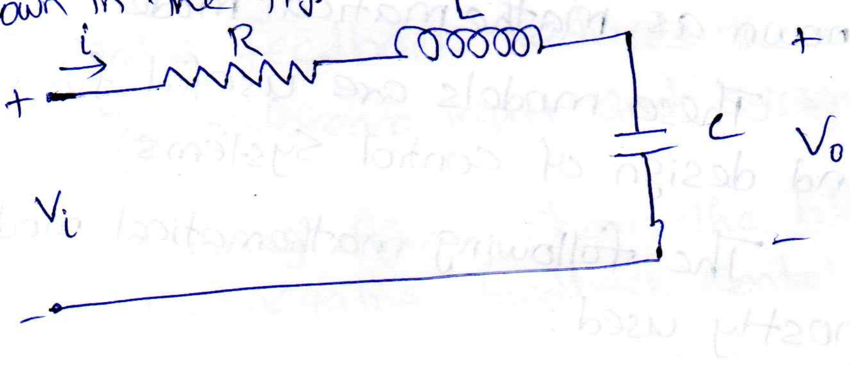
Differential equation model is a time domain mathematical model of control systems.

Follow these steps for differential equation model:

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s):

Example:

Consider the following electrical system as shown in the fig.



This circuit consists of resistor, inductor and capacitor which are connected in series.

Mesh equation for this circuit is,

$$V_i = Ri + L\frac{di}{dt} + V_o$$

Substitute the current passing through the capacitor $i = C.\frac{dV_o}{dt}$

$$\Rightarrow V_i = R.C.\frac{dV_o}{dt} + LC\frac{d^2V_o}{dt^2} + V_o$$

$$\Rightarrow \frac{d^2V_o}{dt^2} + \left(\frac{R}{L}\right)\frac{dV_o}{dt} + \left(\frac{1}{LC}\right)V_o = \left(\frac{1}{LC}\right)V_i$$

The above equation is a second order differential equation.

ii) Transfer function model:

Transfer function model is an s-domain mathematical model of control system.

The transfer function of a linear time

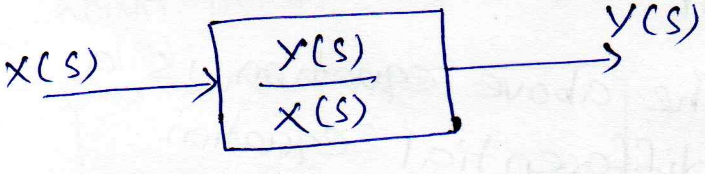
invariant (LTI) system is defined as the ratio of Laplace transform of output and Laplace transform of input by assuming all the initial conditions are zero.

If $x(t)$ and $y(t)$ are the input and output of an LTI system, then the corresponding Laplace transforms are $X(s)$ and $Y(s)$.

Therefore, the transfer function of LTI system

$$= \frac{Y(s)}{X(s)}$$

The transfer function model of LTI system is shown in fig. below



Example:

Previously, we got the differential equation of an electrical system as,

$$\frac{d^2V_0}{dt^2} + \left(\frac{R}{L}\right)\frac{dV_0}{dt} + \left(\frac{1}{LC}\right)V_0 = \left(\frac{1}{LC}\right)V_i$$

Apply Laplace transform on both sides,

$$s^2 V_o(s) + \left(\frac{sR}{L}\right) V_o(s) + \left(\frac{1}{LC}\right) V_o(s) = \left(\frac{1}{LC}\right) V_i(s)$$

$\Rightarrow$

$$\left\{ s^2 + \left(\frac{R}{L}\right) s + \frac{1}{LC} \right\} V_o(s) = \left(\frac{1}{LC}\right) V_i(s)$$

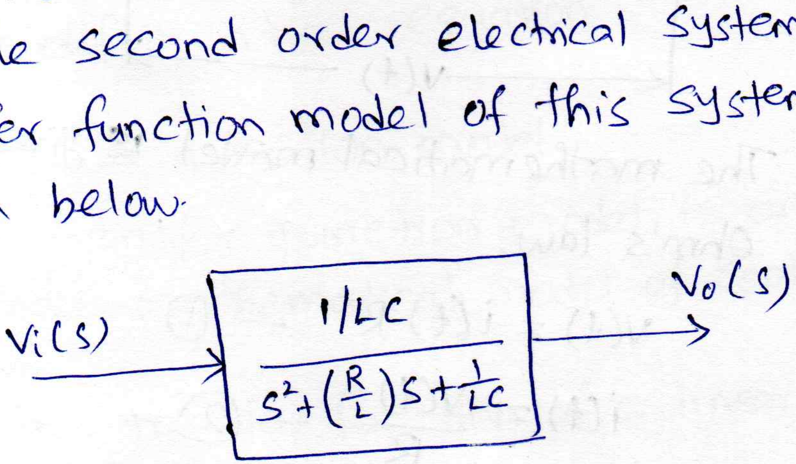
$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{LC}}{s^2 + \left(\frac{R}{L}\right) s + \frac{1}{LC}}$$

where,

$V_i(s) \rightarrow$ Laplace transform of input voltage V_i

$V_o(s) \rightarrow$ Laplace transform of output voltage V_o.

The above equation is a transfer function of the second order electrical system. The transfer function model of this system is shown below.



Mathematical modelling of Electrical and Mechanical systems:

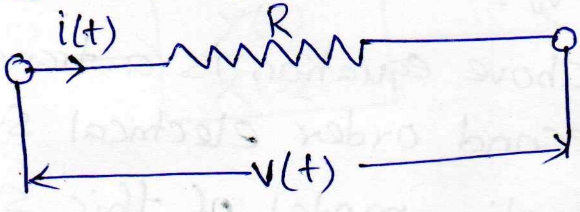
1. Electrical systems:

The 3 basic elements of any electrical system are : resistor, inductor and capacitor.

Circuits consisting of these three elements are analysed by using kirchhoff's Voltage law and Current law.

a) Resistor:

The circuit model of a resistor is as shown. in fig.



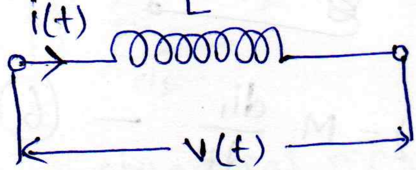
The mathematical model is given by using Ohm's law,

$$v(t) = i(t) R \quad - ①$$

$$i(t) = \frac{v(t)}{R} \quad - ②$$

b) Inductor:

The circuit representation is as shown in fig.



The input-output relationships are given by, Faraday's law,

$$v(t) = L \cdot \frac{di(t)}{dt} \quad - ③$$

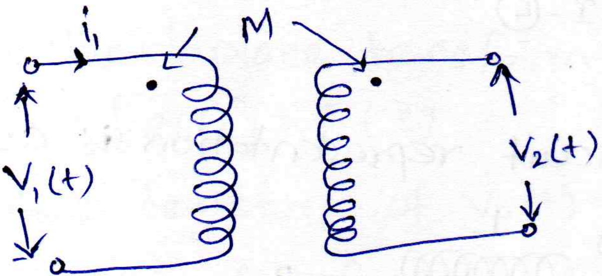
$$i(t) = \frac{1}{L} \int v \, dt \quad - ④$$

where $\int v \, dt$ is known as the flux linkages $\psi(t)$, Thus,

$$i(t) = \frac{\psi(t)}{L} \quad - ⑤$$

If only a single coil is considered, the inductance is known as self inductance.

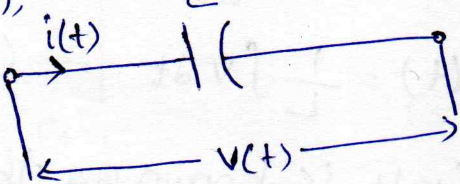
If a voltage is induced in a second coil due to the current in first coil, then this is due to the mutual inductance, as shown in fig.



Here, $V_2(t) = M. \frac{di_1}{dt}$ — ⑥

c) Capacitor:

The circuit symbol of a capacitor is given by,



$v(t) = \frac{1}{c} \int i\, dt$ — ⑦

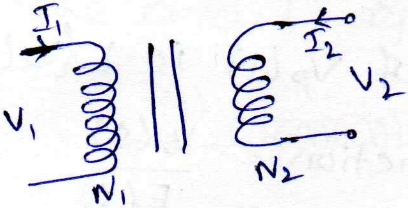
and $i(t) = C. \frac{dv}{dt}$ — ⑧

In ⑦, $\int i\, dt$ is known as the charge on the capacitor and denoted as 'q',

$\therefore q = \int i\, dt$ — ⑨

$\therefore v(t) = \frac{q(t)}{c}$ — ⑩

Another useful element in electrical system is the ideal transformer as shown in fig.



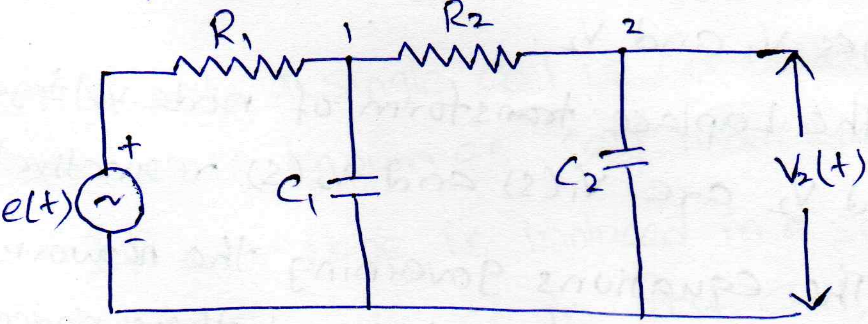
The mathematical model of a transformer is given by,

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} = \frac{I_1}{I_2}$$

Electrical networks consisting of the above elements are analysed using kirchhoff's laws.

Problems:

1) Obtain the transfer function of the electrical network shown in fig.



Soln.:

In the given network, input is e(t) and output is $V_2(t)$.

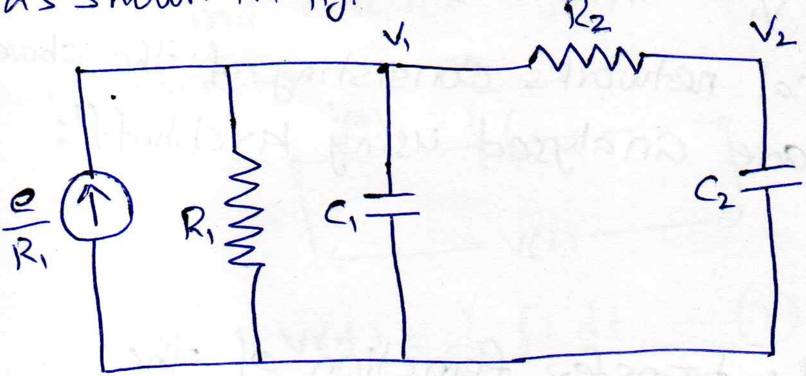
Let the laplace transform of $e(t)$ is

$$L\{e(t)\} = E(s)$$

Laplace transform of $V_2(t)$ is $L\{V_2(t)\} = V_2(s)$

$\therefore$ The transfer function $= \frac{V_2(s)}{E(s)}$

Transform the voltage source in series with resistance into equivalent current source as shown in fig.



The network has two nodes, with node voltages V_1 and V_2.

The Laplace transform of node voltages V_1 and V_2 are $V_1(s)$ and $V_2(s)$ respectively.

The equations governing the network are given by KCL equations at these nodes,

Apply KCL at node 1,

I-⑤

$$\frac{V_1}{R_1} + C_1\frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2} = \frac{e}{R_1}$$

On taking laplace transform of above equation,

$$\frac{V_1(S)}{R_1} + C_1 s V_1(S) + \frac{V_1(S)}{R_2} - \frac{V_2(S)}{R_2} = \frac{E(S)}{R_1}$$

$$V_1(S)\left[\frac{1}{R_1} + SC_1 + \frac{1}{R_2}\right] - \frac{V_2(S)}{R_2} = \frac{E(S)}{R_1} \quad — ①$$

Now, apply KCL at node-2.

$$\frac{V_2 - V_1}{R_2} + C_2\frac{dV_2}{dt} = 0$$

On taking laplace transform,

$$\frac{V_2(S)}{R_2} - \frac{V_1(S)}{R_2} + C_2 S \cdot V_2(S) = 0.$$

$$\frac{V_1(S)}{R_2} = \frac{V_2(S)}{R_2} + C_2 S\, V_2(S)$$

$$= \left[\frac{1}{R_2} + SC_2\right] V_2(S)$$

$$\therefore V_1(S) = \left[1 + SC_2R_2\right] V_2(S) \quad — ②$$

substitute ② in ①, we get,

$$(1+SC_2R_2)V_2(s)\left[\frac{1}{R_1}+SC_1+\frac{1}{R_2}\right]-\frac{V_2(S)}{R_2}=\frac{E(s)}{R_1}$$

$$\left[\frac{(1+SC_2R_2)(R_2+SC_1R_1R_2+R_1)-R_1}{R_1R_2}\right]V_2(S)=\frac{E(S)}{R_1}$$

$$\therefore\ \frac{V_2(S)}{E(s)}=\frac{R_2}{[(1+SC_2R_2)(R_1+R_2+SC_1R_1R_2)-R_1]}$$

Mechanical Translational systems:

The model of mechanical Translational system can be obtained by using 3 basic elements mass, spring and dash-pot.

The weight of the mechanical system is represented by the element mass and it is assumed to be concentrated at the centre of the body.

The elastic deformation of the body can be represented by spring.

The friction existing in rotating mechanical system can be represented by the dash-pot. The dash-pot is a piston moving inside a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system.

The force acting on a mechanical system is governed by Newton's second law of motion. i.e. the sum of forces acting on a body is zero. or the sum of applied forces is equal to the sum of opposing forces on a body.

List of symbols used:

$x \rightarrow$ Displacement, m

$v \rightarrow \frac{dx}{dt} \rightarrow$ velocity, m/sec.

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \rightarrow$ Acceleration, m/sec^2

$f \rightarrow$ applied force, N (Newton)

$f_m \rightarrow$ Opposing force offered by mass of the body, N

$f_k \rightarrow$ Opposing force offered by the elasticity of the body (spring), N

$f_b \rightarrow$ opposing force offered by the friction of the body (dash-pot), N

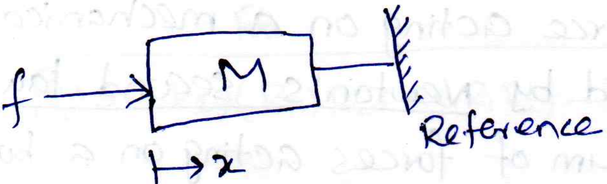
M → mass, kg

K → Stiffness of spring, N/m

B → Viscous friction co-efficient, N-sec/m

i) Mass :

Consider an ideal mass element shown in



Let a force (f) be applied on it.

The mass will offer an opposing force which is proportional to acceleration of the body.

$$f_m \propto \frac{d^2x}{dt^2} \quad \text{(or)} \quad f_m = M.\frac{d^2x}{dt^2}$$

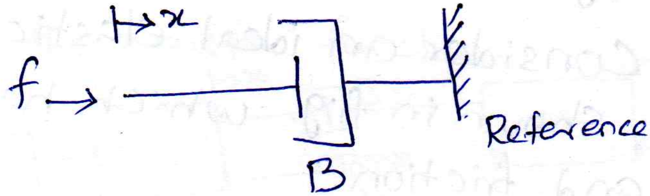
By Newton's second law,

$$f = f_m = M.\frac{d^2x}{dt^2} \qquad \text{— ①}$$

ii) Dash-pot :

Consider an ideal frictional element dash-pot shown in fig., which has negligible mass and elasticity.

I - ⑥



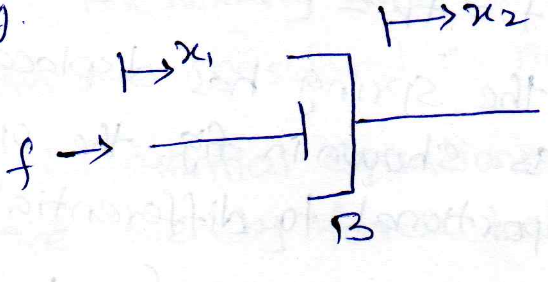
Let a force (f) be applied on it.

$f_b \propto \frac{dx}{dt}$ (or) $f_b = B \cdot \frac{dx}{dt}$

By Newton's second law,

$f = f_b = B \cdot \frac{dx}{dt}$ — ②

When the dash-pot has displacement at both ends as shown in fig., the opposing force is proportional to differential velocity.



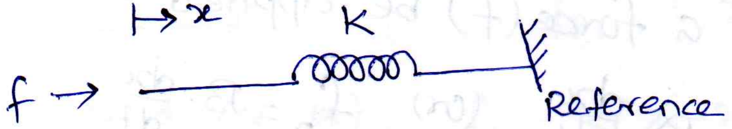
$f_b \propto \frac{d}{dt}(x_1 - x_2)$ (or)

$f_b = B \cdot \frac{d}{dt}(x_1 - x_2)$

$\therefore f = f_b = B \cdot \frac{d}{dt}(x_1 - x_2)$ — ③

iii) Spring:

Consider an ideal elastic element spring shown in fig. which has negligible mass and friction.



Let a force (f) be applied on it. The spring offers an opposing force which is proportial to displacement of the body.

$$f_K \propto x \quad (or) \quad f_K = Kx$$

By Newton's second law.

$$f = f_K = Kx \quad — ④$$

When the spring has displacement at both ends as shown in fig., the opposing force is proportional to differential displacement

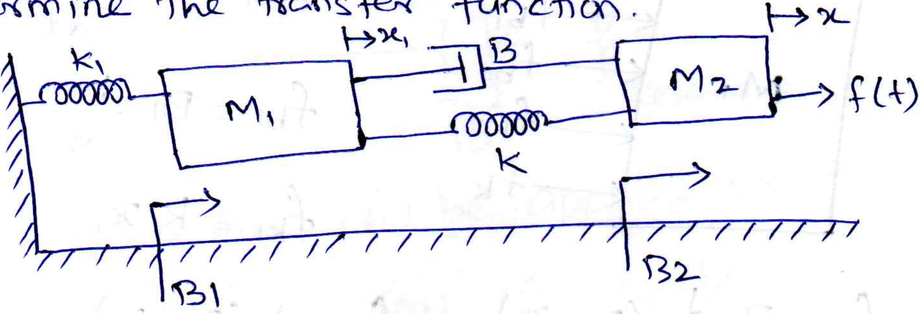
$$f_K \propto (x_1 - x_2) \quad (or) \quad f_K = K(x_1 - x_2)$$

$$\therefore f = f_K = K(x_1 - x_2) \quad — ⑤$$

Problems:

1) Write the differential equations governing

the mechanical system shown in fig. and determine the transfer function.



Soln.:

In the given system, the applied force $f(t)$ is the input and displacement x is the output.

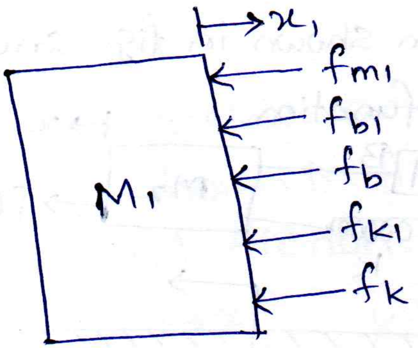
$\therefore$ The transfer function is $\frac{X(s)}{F(s)}$

The system has two nodes and they are M_1 and M_2. (Generally the mass elements in the system are considered as nodes).

The differential equations governing the system are given by force balance equations at these nodes.

The free body diagram of mass M_1 is as shown in fig.

The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1} and f_k.



$$f_{m1} = M_1 . \frac{d^2 x_1}{dt^2}$$

$$f_{b1} = B_1 . \frac{dx_1}{dt}$$

$$f_{k1} = K_1 x_1$$

$$f_b = B . \frac{d}{dt}(x_1 - x), \quad f_k = K(x_1 - x)$$

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 . \frac{dx_1}{dt} + B . \frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

On taking laplace transform with zero initial conditions,

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs[X_1(s) - X(s)] + K_1 X_1(s) + K[X_1(s) - X(s)] = 0$$

$$X_1(s)[M_1 s^2 + (B_1 + B)s + (K_1 + K)] - X(s)[Bs + K] = 0$$

$$X_1(s)[M_1 s^2 + (B_1 + B)s + (K_1 + K)] = X(s)[Bs + K]$$

$$\therefore X_1(s) = X(s) \frac{Bs + K}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} \quad — ①$$

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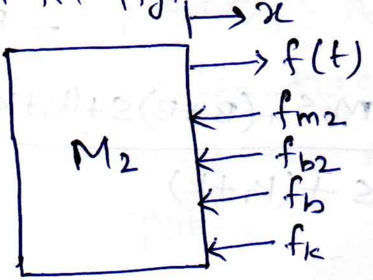
The free body diagram of mass M_2 is shown in fig.



The opposing forces acting on M_2 are f_{m2}, f_{b2}, f_b and f_K.

$f_{m2} = M_2 \cdot \frac{d^2x}{dt^2}$, $f_{b2} = B_2 \cdot \frac{dx}{dt}$, $f_b = B \frac{d}{dt}(x-x_1)$

$f_K = K(x - x_1)$

By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_K = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B \cdot \frac{d}{dt}(x-x_1) + K(x-x_1) = f(t)$$

On taking laplace transform of above equation,

$$M_2 S^2 X(S) + B_2 S X(S) + BS[X(S) - X_1(S)] + K[X(S) - X_1(S)] = F(S)$$

$$X(S)[M_2 S^2 + (B_2 + B)S + k] - X_1(S)[BS + K] = F(S) \quad -②$$

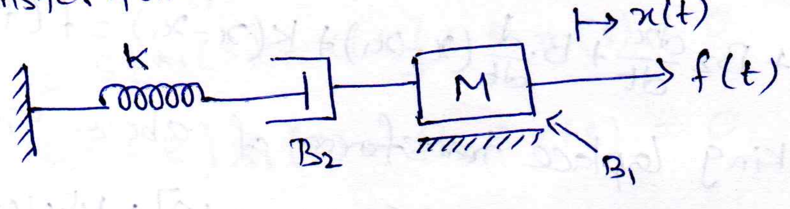
Substituting $X_1(S)$ from ① in ②,

$$X(S)\left[M_2 S^2+(B_2+B)S+k\right]-X(S)\frac{(Bs+k)^2}{M_1S^2+(B_1+B)S+(k_1+k)} = F(S)$$

$$X(S)\left[\frac{\left[M_2 s^2+(B_2+B)S+k\right]\left[M_1S^2+(B_1+B)S+(k_1+k)\right]-(Bs+k)^2}{M_1 S^2+(B_1+B)S+(k_1+k)}\right] = F(S)$$

$$\therefore \frac{X(S)}{F(S)} = \frac{M_1S^2+(B_1+B)S+k_1+k}{\left[M_2 S^2+(B_2+B)S+k\right]\left[M_1S^2+(B_1+B)S+(k_1+k)\right]-(Bs+k)^2}$$

2) Write the equations of motion in s-domain for the system shown in fig. Determine the transfer function of the system.



Soln.:

Let laplace transform of $x(t)$ =

$L\{x(t)\} = X(S)$

Laplace transform of $f(t) = L\{f(t)\} = F(S)$

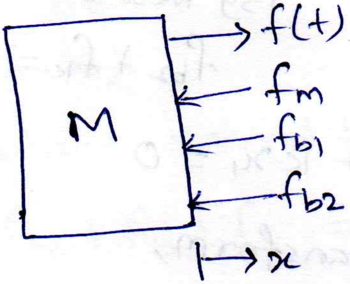
Let x_1 be the displacement at the

meeting point of spring and dashpot

$$L\{x_1(t)\} = X_1(s)$$

The system has two nodes and they are mass M and the meeting point of spring and dashpot.

The free body diagram of mass M is shown in fig. The opposing forces are marked as fm, fb1, and fb2.



$$f_m = M.\frac{d^2x}{dt^2} \quad ; \quad f_{b1} = B_1\frac{dx}{dt}$$

$$f_{b2} = B_2\frac{d}{dt}(x - x_1)$$

By Newton's second law,

$$f_m + f_{b1} + f_{b2} = f(t)$$

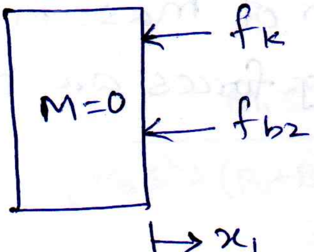
$$\therefore M.\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d}{dt}(x - x_1) = f(t)$$

Taking laplace transform,

$$Ms^2X(s) + B_1 s X(s) + B_2 s[X(s) - X_1(s)] = F(s)$$

$$[Ms^2 + (B_1+B_2)s]X(s) - B_2 s X_1(s) = F(s) \quad \text{— ①}$$

The free body diagram at the meeting point of spring and dashpot as shown in fig. The opposing forces are marked as f_K and f_{b2}.



$$f_{b2} = B_2 \cdot \frac{d}{dt}(x_1 - x)$$

$$f_K = k x_1$$

By newton's law,

$$f_{b2} + f_K = 0$$

$$B_2 \cdot \frac{d}{dt}(x_1 - x) + k x_1 = 0$$

Taking laplace transform,

$$B_2 \cdot s[X_1(s) - X(s)] + K X_1(s) = 0$$

$$(B_2 s + K)X_1(s) - B_2 s X(s) = 0$$

$$\therefore X_1(s) = \frac{B_2 s}{B_2 s + K} X(s) \quad \text{— ②}$$

Substituting for $X_1(s)$ from ② in ①

$$[Ms^2 + (B_1+B_2)s]X(s) - B_2 s\left[\frac{B_2 s}{B_2 s + K}\right]X(s) = F(s)$$

$$X(s)\frac{\left[[Ms^2+(B_1+B_2)s](B_2s+K)-(B_2s)^2\right]}{B_2s+K}=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{B_2s+K}{[Ms^2+(B_1+B_2)s](B_2s+K)-(B_2s)^2}$$

Mechanical Rotational Systems:

The basic 3 elements of mechanical rotational systems are, moment of inertia (J) of mass, dash-pot with rotational frictional co-efficient (B) and torsional spring with stiffness (K).

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system.

The torques acting on a rotational mechanical body are governed by <u>Newton's second law of motion</u> for rotational systems

It states that, the sum of torques acting on a body is zero. (or) the sum of applied torques is equal to the sum of opposing torques on a body.

List of symbols used:

$\theta \rightarrow$ Angular displacement, rad

$\frac{d\theta}{dt} \rightarrow$ Angular velocity, rad/sec.

$\frac{d^2\theta}{dt^2} \rightarrow$ Angular acceleration, rad/sec^2

$T \rightarrow$ Applied torque, N-m

$J \rightarrow$ Moment of inertia, kg-m^2/rad

$B \rightarrow$ rotational frictional co-efficient, N-m/(rad/sec)

$K \rightarrow$ Stiffness of the spring N-m/rad

1) Moment of inertia:

Consider an ideal mass element shown in fig., which has negligible friction and elasticity.

The opposing torque due to moment of

inertia is proportional to the angular acceleration.

Let $T_j \rightarrow$ opposing torque due to moment of inertia of the body.

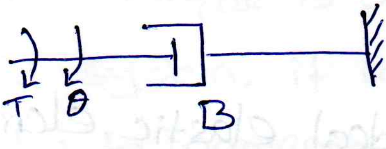
$$T_j \propto \frac{d^2\theta}{dt^2} \quad , \quad T_j = J.\frac{d^2\theta}{dt^2}$$

By ~~netw~~ newton's second law,

$$T = T_j = J.\frac{d^2\theta}{dt^2}$$

2) Dash-pot :

Consider an ideal frictional element dash pot shown in fig. which has negligible moment of inertia and elasticity.



The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

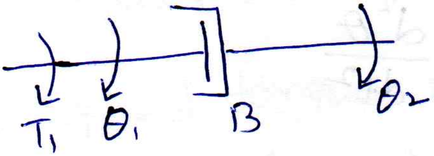
Let $T_b \rightarrow$ opposing torque due to friction.

$$T_b \propto \frac{d\theta}{dt} \quad , \quad T_b = B.\frac{d\theta}{dt}$$

Newton's second law,

$$T = T_b = B \cdot \frac{d\theta}{dt}$$

When the dash pot has angular displacement at both ends as shown in fig., the opposing torque is proportional to the differential angular velocity.

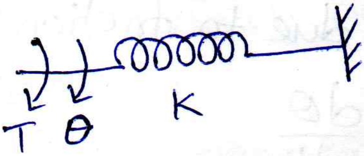


$$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2) \quad , \quad T_b = B \cdot \frac{d}{dt}(\theta_1 - \theta_2)$$

$$\therefore T = T_b = B \cdot \frac{d}{dt}(\theta_1 - \theta_2)$$

3) Spring:

Consider an ideal elastic element, torsional spring as shown in fig. which has negligible moment of inertia and friction.



The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

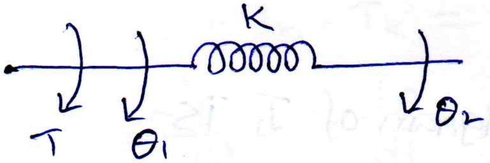
Let. $T_k \rightarrow$ Opposing torque due to elasticity

$T_k \alpha \theta$, $T_k = K\theta$

By Newton's second law,

$$T = T_k = K\theta$$

When the spring has angular displacement at both ends as shown in fig., the opposing torque is ~~po~~ Proportional to differential angular displacement.



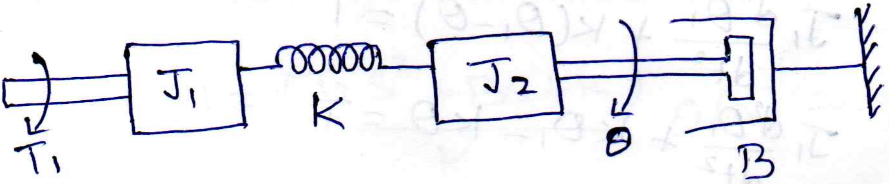
$T_k \alpha (\theta_1 - \theta_2)$

$T_k = K(\theta_1 - \theta_2)$

$$\therefore T = T_k = k(\theta_1 - \theta_2)$$

Problems:

1) Write the differential equations governing the mechanical rotational system shown in fig. Obtain the transfer function of the system.



Soln.:

Applied torque T is the input

angular displacement θ is the output

$$L[T] = T(s)$$

$$L[\theta] = \theta(s)$$

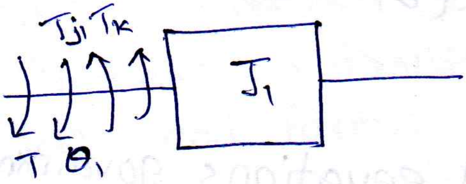
$$\text{Transfer function} = \frac{\theta(s)}{T(s)}$$

The system has two nodes (J_1 and J_2).

Let, the angular displacement of mass with moment of inertia J_1 be θ_1.

$$L[\theta_1] = \theta_1[s]$$

The free body diagram of J_1 is



$$T_{J1} = J_1 . \frac{d^2\theta_1}{dt^2}$$

$$T_K = K(\theta_1 - \theta)$$

By Newton's law,

$$T_{j1} + T_K = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

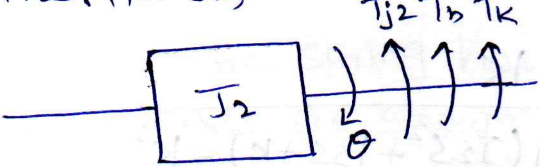
$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$$

Taking laplace transform on both sides,

$$J_1 S^2\ \theta_1(S) + K\ \theta_1(S) - K\ \theta(S) = T(S)$$

$$(J_1 S^2 + K)\ \theta_1(S) - K\ \theta(S) = T(S) \quad — ①$$

The freebody diagram of mass with moment of inertia J_2,



$$T_{j2} = J_2 . \frac{d^2\theta}{dt^2} \qquad T_b = B . \frac{d\theta}{dt} \qquad T_K = K(\theta - \theta_1)$$

$$\Rightarrow T_{j2} + T_b + T_K = 0$$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B . \frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B . \frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

Taking laplace transform,

$$J_2 . S^2\ \theta(S) + BS\ \theta(S) + K\ \theta(S) - K\theta_1(S) = 0$$

$$(J_2 S^2 + BS + K)\ \theta(S) - K\ \theta_1(S) = 0$$

$$\theta_1(S) = \frac{(J_2 S^2 + BS + K)}{K}\ \theta(S) \quad — ②$$

Substitute ② in ①, we get,

$$(J_1 s^2 + k) \frac{(J_2 s^2 + Bs + k)}{k} \theta(s) - k\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + k)(J_2 s^2 + Bs + k) - k^2}{k} \right] \theta(s) = T(s)$$

$$\Rightarrow \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + k)(J_2 s^2 + Bs + k) - k^2}$$

Analogy between electrical and mechanical systems:

Systems remain analogous as long as the differential equations governing the systems or transfer functions are of identical form.

The three basic elements mass, dash-pot and spring that are used in modelling mechanical translational system are analogous to resistance, inductance and capacitance of electrical systems.

The input force in mechanical system is analogous to either voltage source or current source in electrical systems.

The output velocity (first order derivative of displacement) in mechanical system is analogous to either current or voltage in an element in electrical system.

Since, the electrical systems has two types of inputs either voltage or current source, there are two types of analogies:

a) Force - voltage analogy

& b) Force - current analogy.

a) Force - voltage analogy:

The analogous electrical and mechanical quantities are tabulated in the below table.

Electrical system	Mechanical systems	
	Translational	Rotational.
Voltage v	Force f	Torque T
current i	Velocity v	Angular velocity ω
charge q	Displacement x	Angular displacement θ

Inductance L	Mass M	Moment of inertia J
Capacitance C	Compliance $\frac{1}{K}$	Compliance $\frac{1}{K}$
Resistance R	Damping Co-efficient B	Damping Co-efficient B

b) Force-current analogy:

The analogous quantities based on force-current analogy are given in table below.

Electrical system.	Mechanical System	
	Translational	Rotational.
Current i	Force f	Torque T
Voltage v	velocity v	angular velocity ω
Flux linkages ψ	Displacement x	angular displacement θ
Capacitance C	Mass M	Moment of inertia J
Conductance G	Damping Coefficient B	Rotational damping coefficient B
Inductance L	Compliance $\frac{1}{K}$	Compliance $\frac{1}{K}$

Block diagram reduction techniques:

Introduction:

A control system may consists of a number of components.

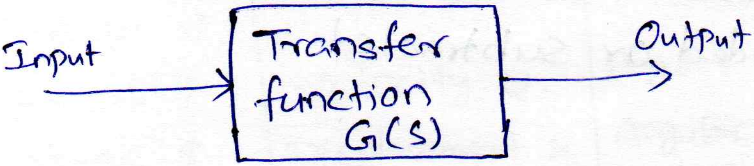
A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.

The basic elements of a block diagram are block, branch point and summing point.

1) Block :

The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.

The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals.



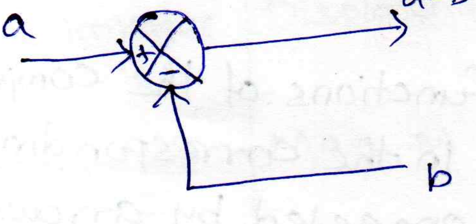
The output signal from the block is given by the product of input signal and transfer function in the block.

2) Branch point :

A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.

3) Summing Point :

Summing points are used to add two or more signals in the system.



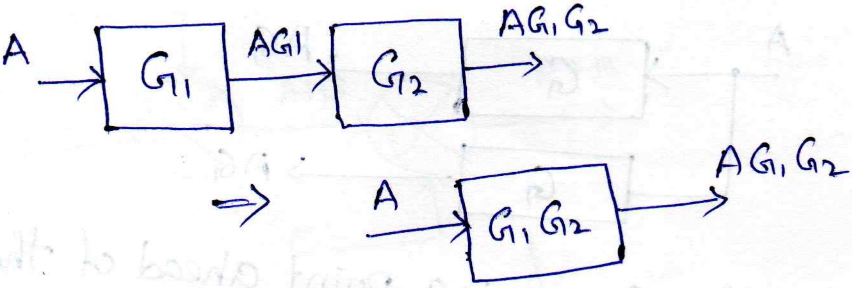
The plus or minus sign at each arrowhead indicates whether the signal is to be added or subtracted.

Block diagram reduction:

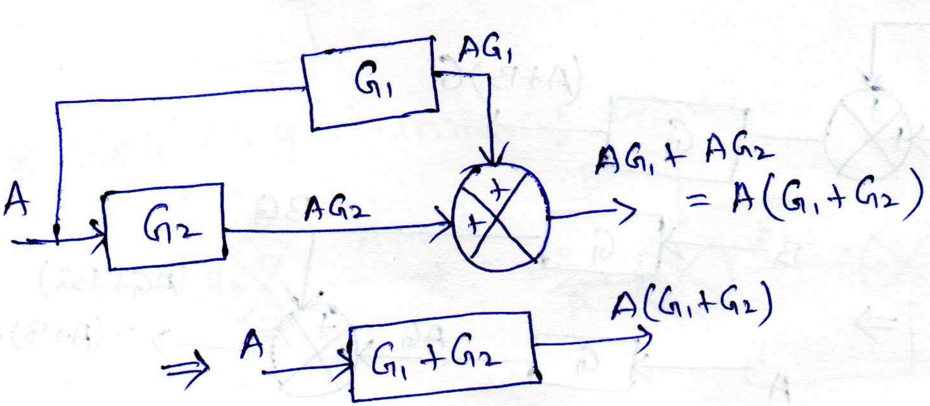
The block diagram can be reduced to find the overall transfer function of the system.

The following rules can be used for block diagram reduction which provides the modifications without altering the input output relations.

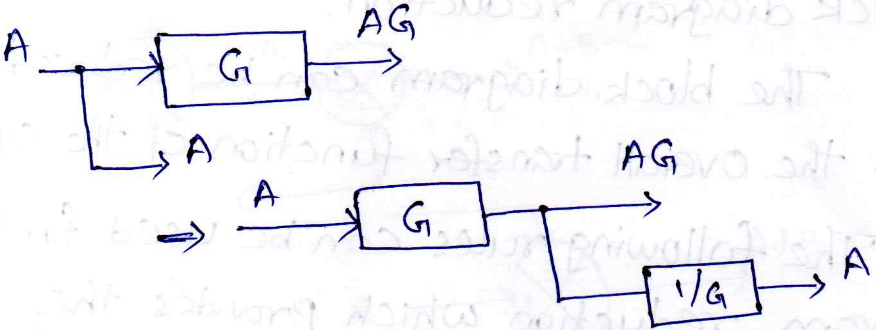
1. Combining the blocks in cascade:



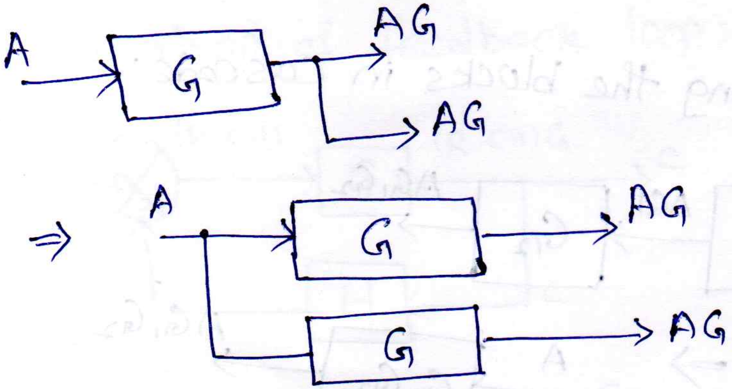
2. Combining parallel blocks



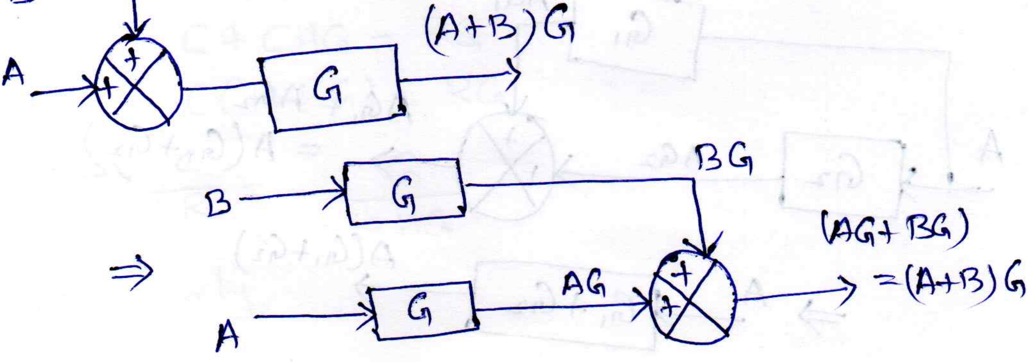
3. Moving the branch point ahead of the block.



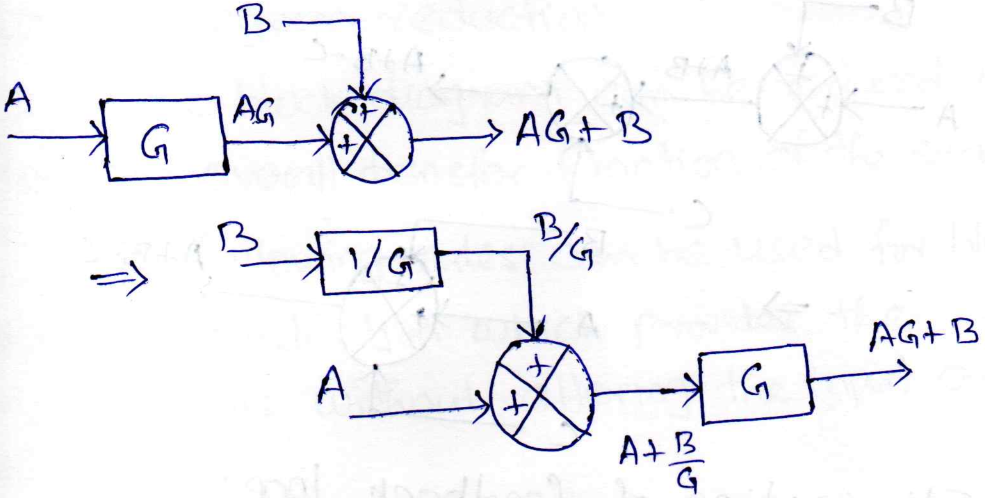
4. Moving the branch point before the block.



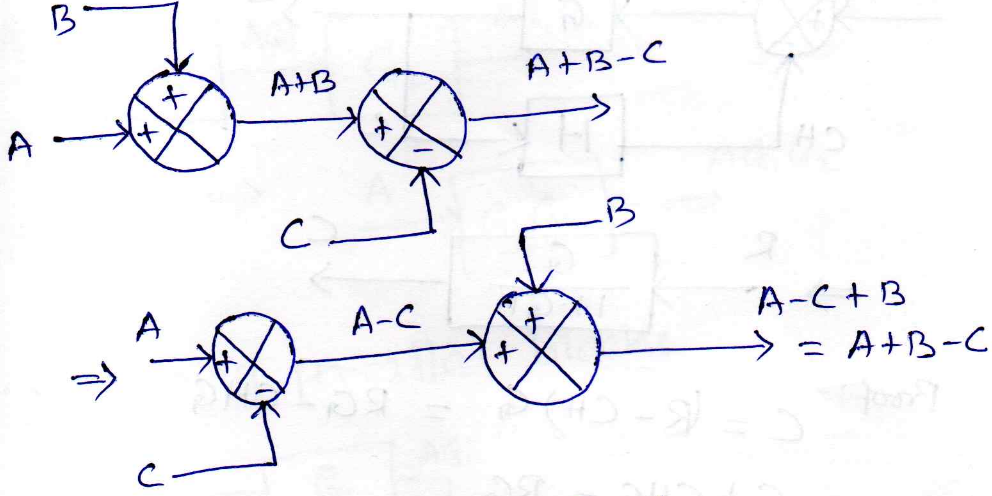
5. Moving the summing point ahead of the block.



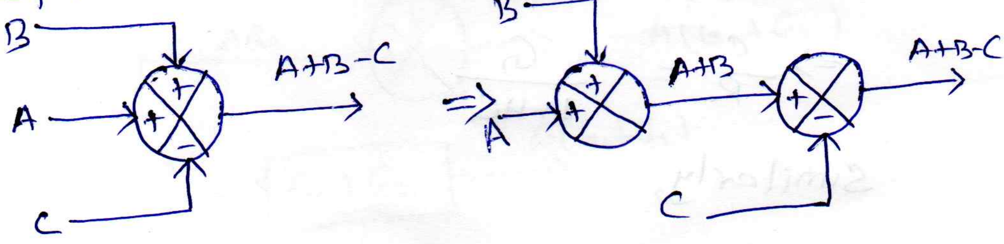
6. Moving the summing point before the block.



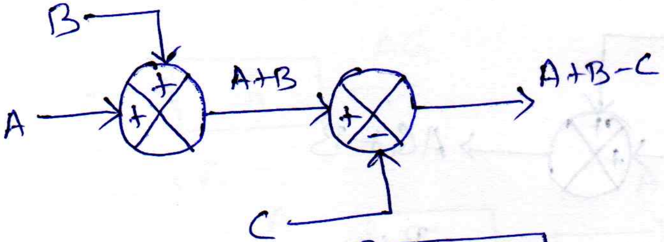
7. Interchanging summing point :



8. Splitting summing points



9. Combining summing points



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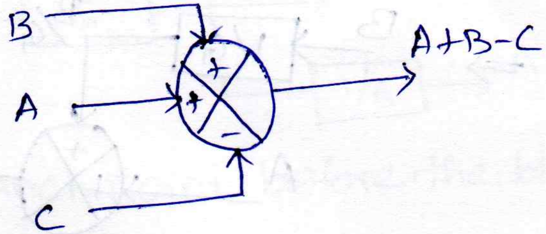
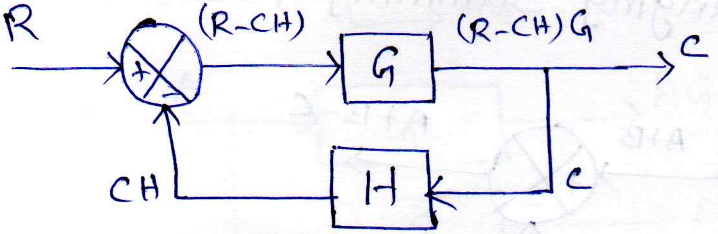
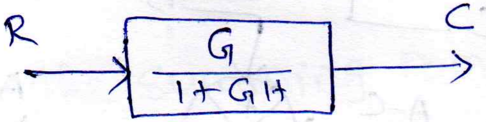


10. Elimination of feedback loop:



⇒



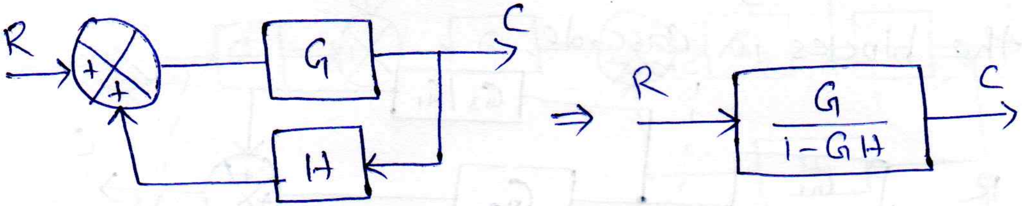
Proof: $C = (R - CH)\,G = RG - CHG$

$C + CHG = RG$

$C(1 + HG) = RG.$

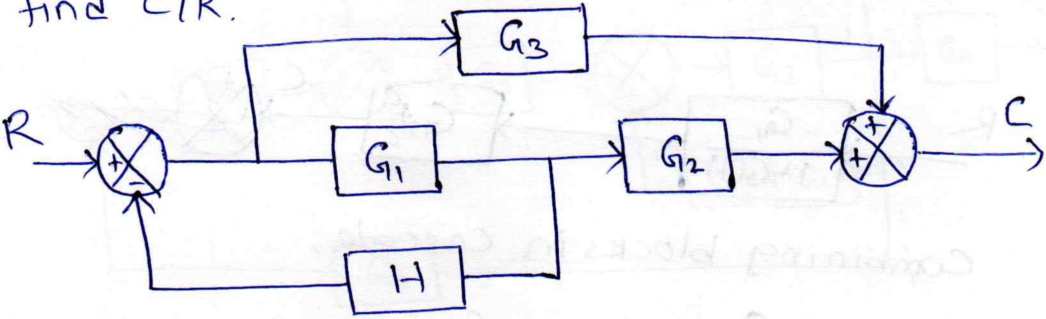
$\Rightarrow \frac{C}{R} = \frac{G}{1+HG}$

Similarly,



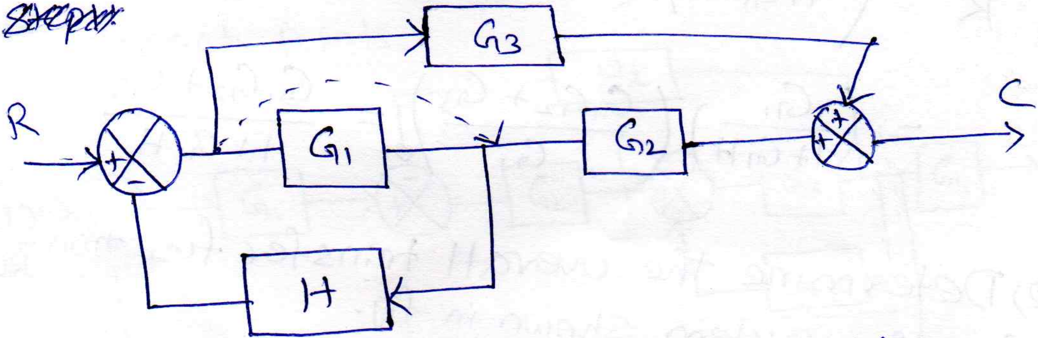
Problems:

1) Reduce the block diagram shown in fig. and find C/R.

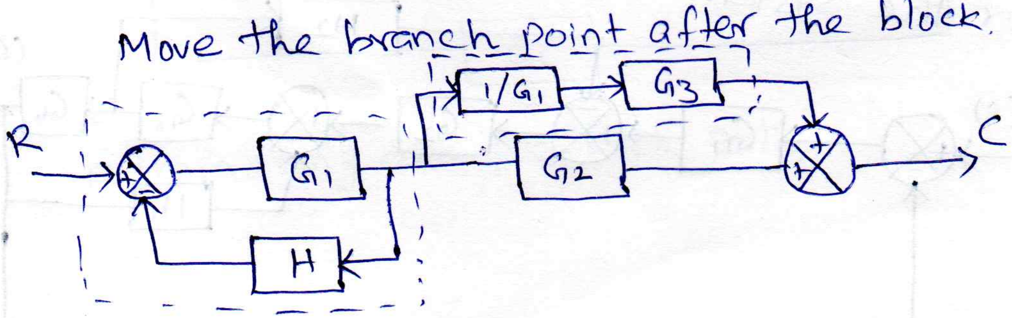


Soln.:

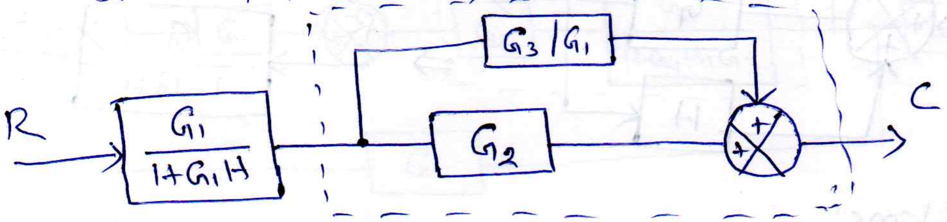
~~Steps~~



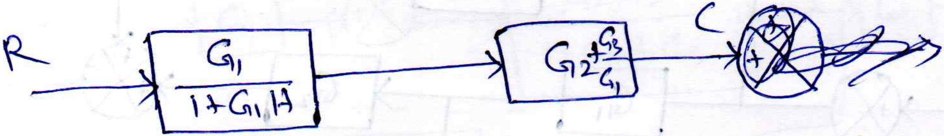
Move the branch point after the block.



Eliminating the feed back path and combining the blocks in cascade,



Combining parallel blocks,

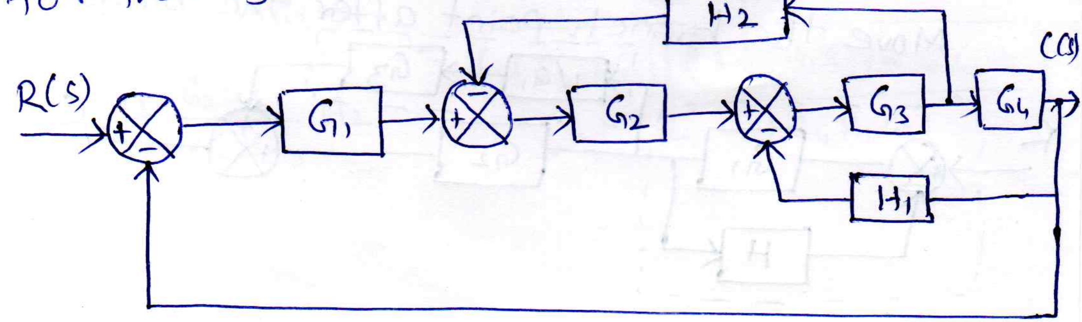


Combining blocks in cascade,

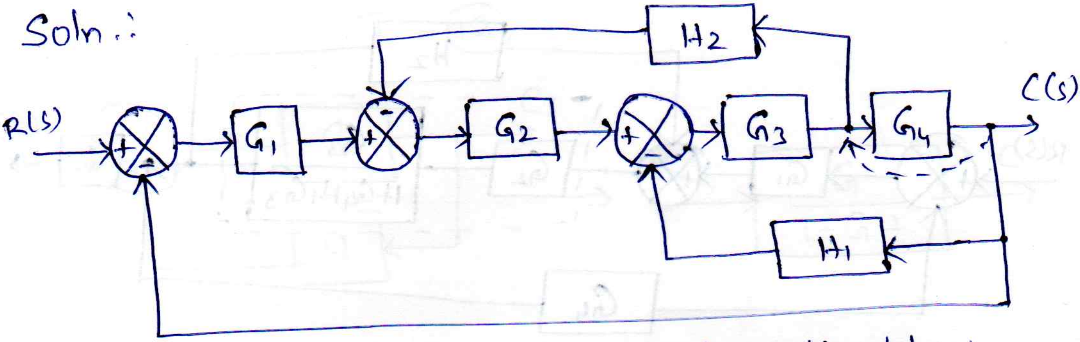
$$\frac{C}{R} = \left(\frac{G_1}{1+G_1H}\right)\left(G_2 + \frac{G_3}{G_1}\right)$$

$$= \left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right) = \frac{G_1G_2+G_3}{1+G_1H}$$

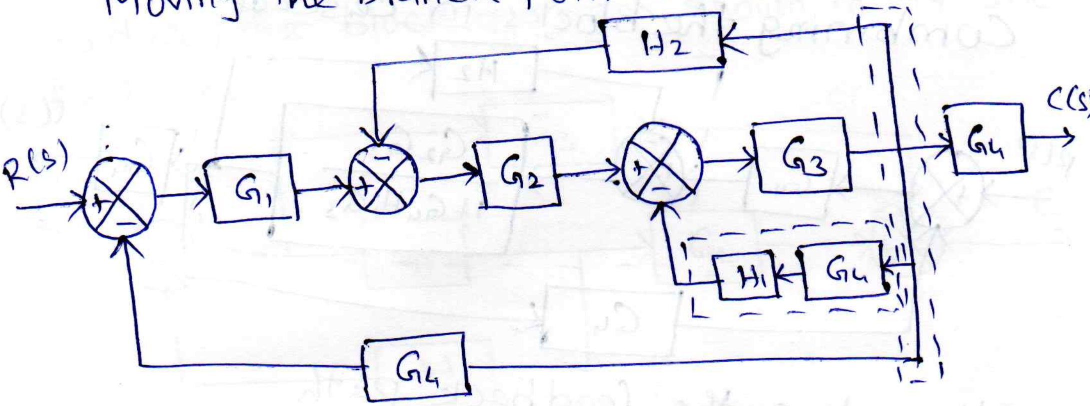
2) Determine the overall transfer function $\frac{C(s)}{R(s)}$ for the system shown in fig.



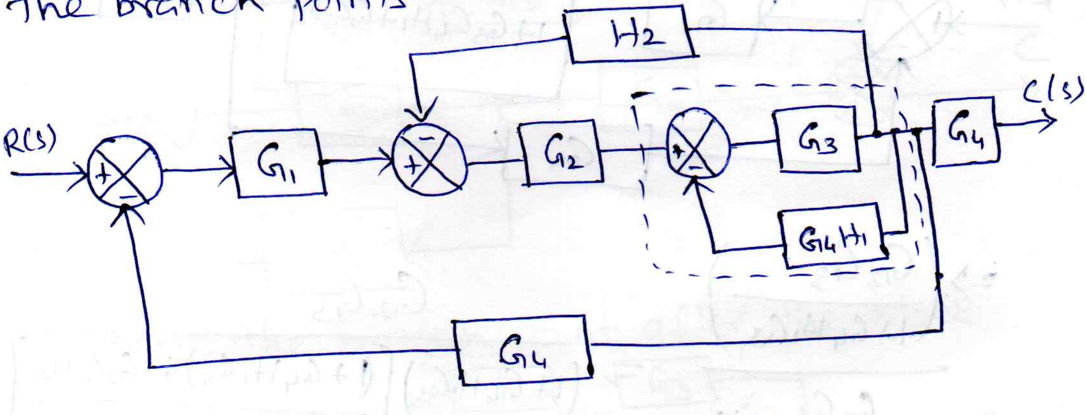
Soln.:



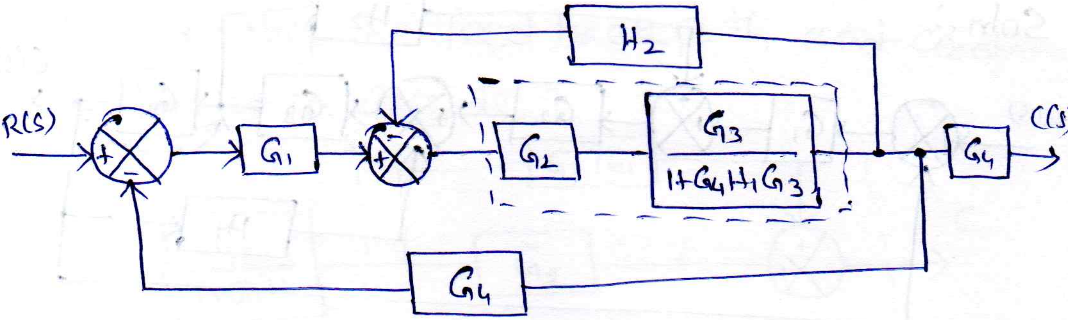
Moving the branch point before the block



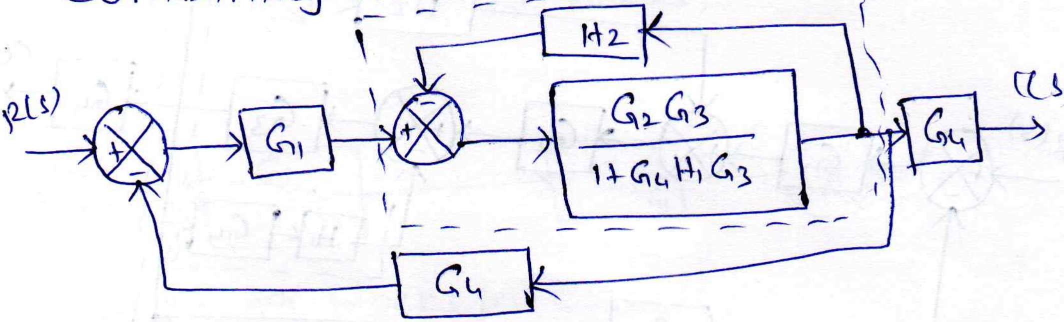
Combining the blocks in cascade and rearranging the branch points



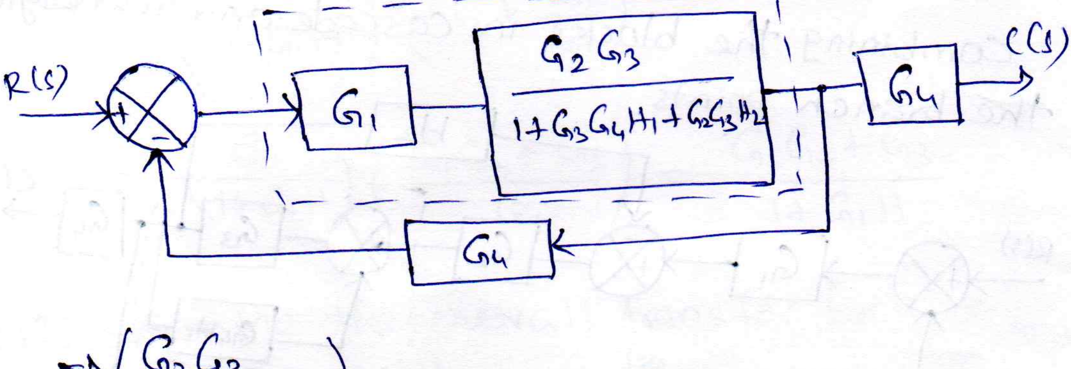
Eliminating the feedback path,



Combining the blocks in cascade



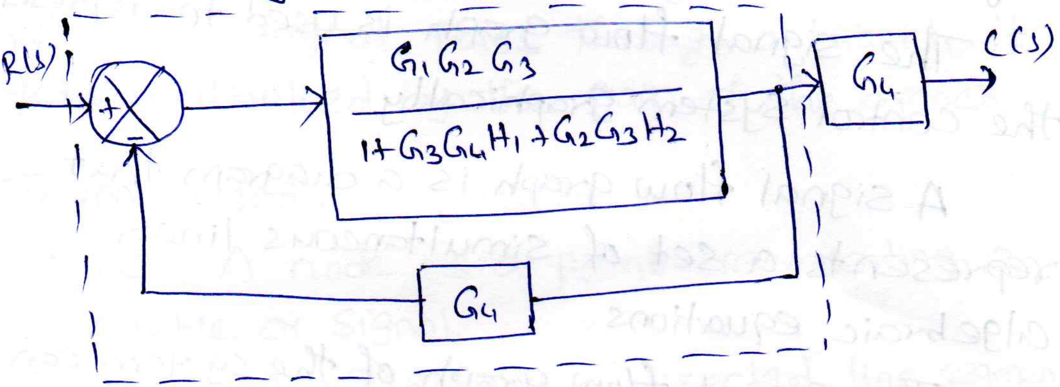
Eliminating the feedback path



$$\Rightarrow \frac{\left(\frac{G_2G_3}{1+G_4H_1G_3}\right)}{1+\frac{G_2G_3}{1+G_4H_1G_3}H_2} = \frac{G_2G_3}{(1+G_4H_1G_3)\left[\frac{(1+G_4H_1G_3)+G_2G_3H_2}{(1+G_4H_1G_3)}\right]}$$

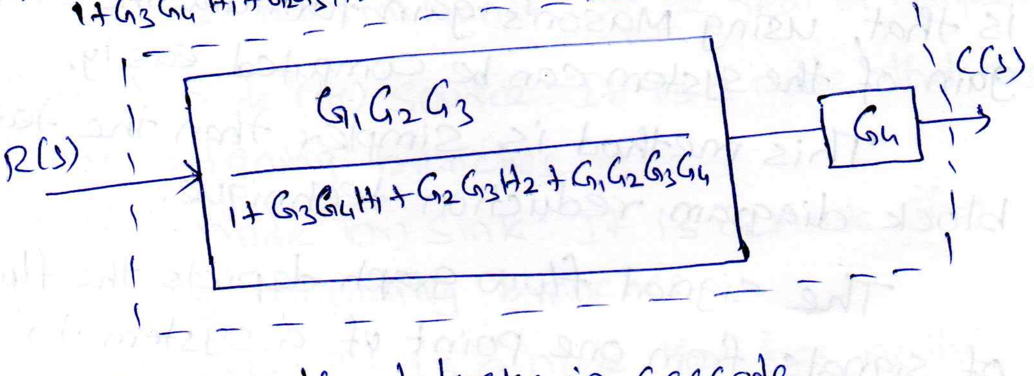
$$= \frac{G_2 G_3}{1+G_4 G_3 H_1 + G_2 G_3 H_2}$$

Combining the blocks in cascade,



Eliminating the feedback path.

$$\frac{\left(\frac{G_1 G_2 G_3}{1+G_3 G_4 H_1 + G_2 G_3 H_2}\right)}{1+\frac{G_1 G_2 G_3}{1+G_3 G_4 H_1 + G_2 G_3 H_2} \times G_4}$$



Combining the blocks in cascade,

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4}$$

Signal flow graph:

The signal flow graph is used to represent the control system graphically.

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations.

The signal flow graph of the system can be constructed using these equations.

It should be noted that the signal flow graph approach and the block diagram approach yield the same information.

The advantages in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed easily.

This method is simpler than the tedious block diagram reduction technique.

The signal flow graph depicts the flow of signals from one point of a system to another and gives relationships among the signals.

A signal flow graph consists of a network in which nodes are connected by directed branches.

Each branch has a gain or transmittance When the signal pass through a branch, it gets multiplied by the gain of the branch.

Terms used:

1) Node: A node is a point representing a variable or signal.

2) Branch: A branch is a directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow.

3) Transmittance: The gain acquired by the signal when it travels from one node to another is called transmittance.

The transmittance can be real or complex.

4) Input node (or) source: It is a node that has only outgoing branches.

5) Output node (or) sink: It is a node that has only incoming branches.

6) Mixed node: It is a node that has both incoming and outgoing branches.

7) Path : A path is a traversal of connected branches in the direction of the branch arrows.

8) Open path: A open path starts at a node and ends at another node.

9) Closed path : Closed path starts and ends at same node.

10) Forward path: It is a path from an input node to an output node that does not cross any node more than once.

11) Forward path gain : It is the product of the branch transmittances (gains) of a forward Path.

12) Loop gain : It is the product of the branch transmittances (gains) of a loop.

13) Individual loop : It is a closed path starting from a node and after passing through a certain part of a graph arrives at the same node without crossing any node more than once.

14) Non-touching loops: If the loops does not have a common node then they are said to be non-touching loops.

Properties of signal flow graph:

1) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.

2) Signal flow graph is applicable only to the linear systems.

3) A node in the signal flow graph represents the variable or signal.

4) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.

5) A branch indicates functional dependence of one signal on the other.

6) The signal travels along branches, only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

7) The signal flow graph of system is not unique, By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

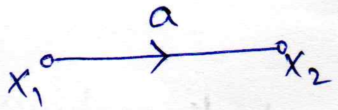
Rules followed for reduction of signal flow graph:

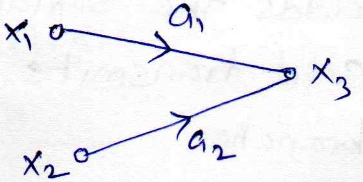
Signal flow graph for a system can be reduced to obtain the transfer function of the system using the following rules.

Rule 1:

Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

Example:





$x_2 = a x_1$

$x_3 = a_1 x_1 + a_2 x_2$

Rule 2:

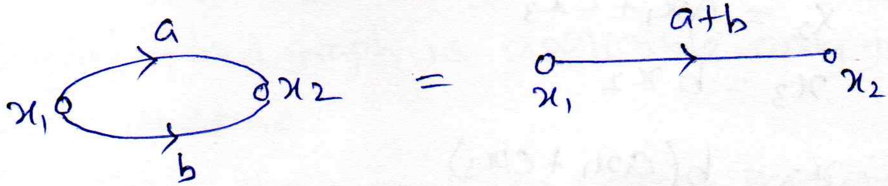
Cascaded branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Example:

Rule 3:

Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

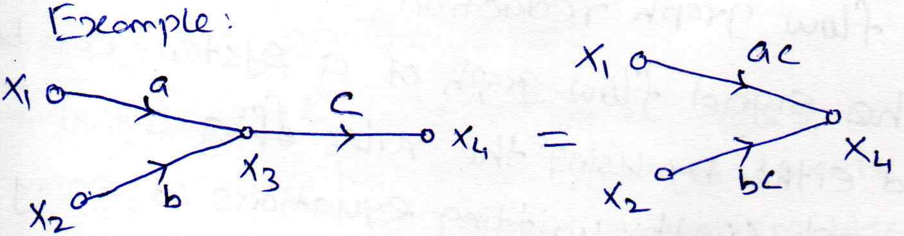
Example:



Rule 4:

A mixed node can be eliminated by multiplying the transmittance of outgoing branch to the transmittance of all incoming branches to the mixed node.

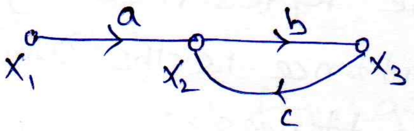
Example:



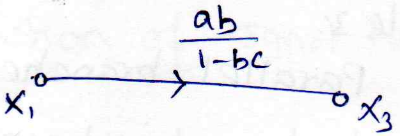
Rule 5:

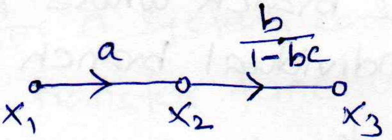
A loop may be eliminated by writing equations at the input and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

Example:



⇒





Proof:

$$x_2 = ax_1 + cx_3$$

$$x_3 = bx_2$$

$$\therefore x_3 = b(ax_1 + cx_3)$$

$$x_3 = abx_1 + bcx_3$$

$$x_3 - bcx_3 = abx_1$$

$$x_3(1-bc) = abx_1$$

$$\frac{x_3}{x_1} = \frac{ab}{1-bc}$$

Signal flow graph reduction:

The signal flow graph of a system can be reduced either by using the rules ~~of~~ for a signal flow graph. (i.e.) by writting equations at every node and then rearranging these equations to get the ratio of output and input (transfer function).

The signal flow graph reduction by above method will be time consuming and tedious.

S.J. Mason has developed a simple procedure to determine the transfer function of the system represented as a signal flow graph, which has been called as Mason's gain formula.

Mason's Gain formula:

It is used to determine the transfer function of the system from the signal flow graph.

~~Let~~ Mason's gain formula states the overall gain of the system as follows,

$$\text{Overall gain } T = \frac{1}{\Delta}\sum_{K} P_K \Delta_K$$

where, $T = T(s) =$ Transfer function of the system

$P_K \rightarrow$ forward path gain of K^{th} forward path

$\Delta = 1 -$ (sum of individual loop gains)

$+ \left(\begin{array}{l}\text{sum of gain products of all possible}\\ \text{combinations of two non-touching loops}\end{array}\right)$

$- \left(\begin{array}{l}\text{sum of gain products of all possible}\\ \text{combinations of three non-touching loops}\end{array}\right)$

$+ \cdots\cdots$

$\Delta_K \rightarrow \Delta$ for that part of the graph which is not touching K^{th} forward path.

Procedure for converting block diagram to signal flow graph:

The signal flow graph and block diagram of a system provides the same information.

Since, the block diagram reduction technique is tedious, it will be easier if the block diagram is converted into signal flow graph and Mason's gain formula is used to find the transfer function.

The following procedure can be used to convert block diagram to signal flow graph.

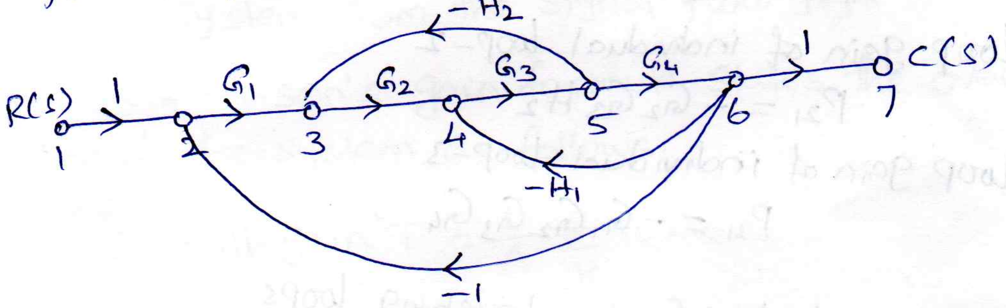
1. Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
2. Draw the nodes seperately as small circles and number them in the order 1,2,3,4,... etc.
3. From the block diagram, find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
4. Draw the feed forward paths between various nodes and mark the gain of feed

forward path along with sign.

5. Draw the feedback paths between various nodes and mark the gain of feedback paths along with sign.

Problems:

1) Find the overall gain $\frac{C(s)}{R(s)}$ for the signal flow graph shown in fig.

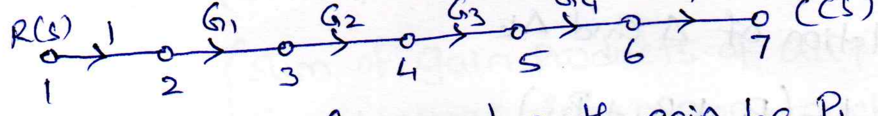


Soln.:

1. Forward path gains

There is only one forward path.

$\therefore K = 1$



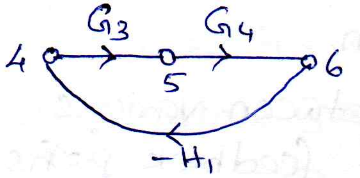
Let the forward path gain be P_1

$$P_1 = G_1 G_2 G_3 G_4.$$

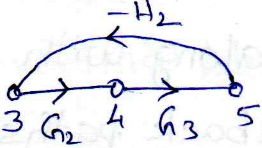
2. Individual loop gain:

There are 3 individual loops.

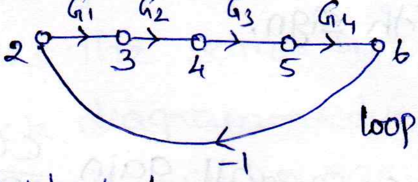
Let the loop gains be P_{11}, P_{21}, P_{31}



Loop 1



loop 2



loop 3

loop gain of individual loop-1.

$$P_{11} = -G_3G_4H_1$$

Loop gain of individual loop-2

$$P_{21} = -G_2G_3H_2$$

Loop gain of individual loop-3

$$P_{31} = -G_1G_2G_3G_4$$

3. Gain products of Non-touching loops

There are no possible combinations of two non-touching loops, three non touching loops, etc.

4. Calculation of Δ and Δ_k.

$$\Delta = 1-(P_{11}+P_{21}+P_{31})$$

$$= 1-(-G_3G_4H_1 - G_2G_3H_2 - G_1G_2G_3G_4)$$

$$= 1+G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4$$

Since, no part of the graph is non-touching with forward path-1,

$$\Delta_1 = 1$$

5- Transfer function, T

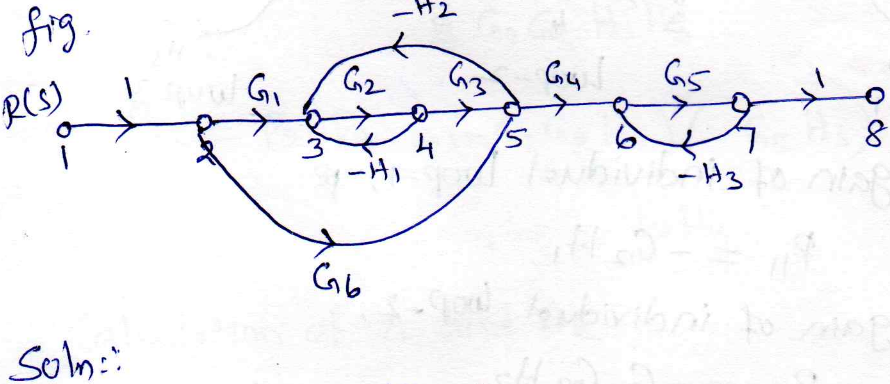
By Mason's gain formula,

$$\frac{C(s)}{R(s)} = \frac{1}{\Delta} \sum_{K} P_K \Delta_K$$

$$= \frac{1}{\Delta} P_1 \Delta_1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4}$$

2) Find the overall transfer function of the system whose signal flow graph is shown in fig.

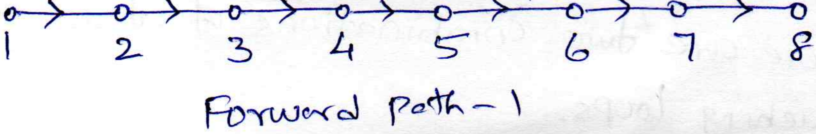


Soln::

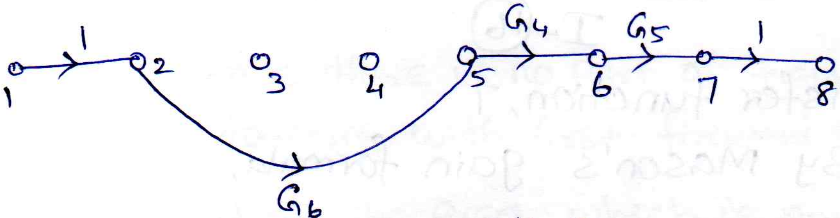
1. Forward path gains:

There are two forward paths $\therefore K = 2$

Let forward path gains be P_1 and P_2



Forward path - 1



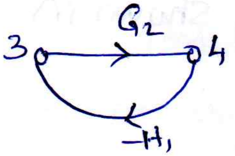
Forward path-2

Gain of forward path-1, $P_1 = G_1 G_2 G_3 G_4 G_5$

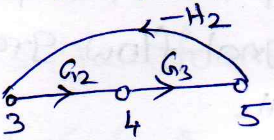
Gain of forward path-2, $P_2 = G_4 G_5 G_6$

2. Individual loop gain:

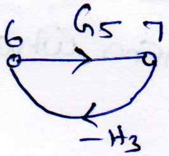
There are 3 individual loops. Let individual loop gains be P_{11}, P_{21}, P_{31}



Loop-1



Loop-2



Loop 3

Loop gain of individual loop-1, ~~P~~

$$P_{11} = -G_2 H_1$$

Loop gain of individual loop-2,

$$P_{21} = -G_2 G_3 H_2$$

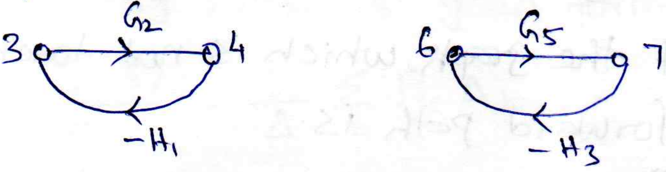
Loop gain of individual loop-3

$$P_{31} = -G_5 H_3$$

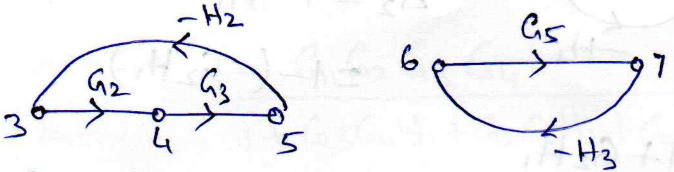
3. Gain products of two non-touching loops:

There are two combinations of two non-touching loops.

Let the gain products of two non touching loops be P_{12} and P_{22}.



First combination of two non-touching loops.



Second combination of two non-touching loops.

$$\therefore P_{12} = P_{11} P_{31} = (-G_2 H_1)(-G_5 H_3)$$

$$= G_2 G_5 H_1 H_3$$

$$P_{22} = P_{21} P_{31} = (-G_2 G_3 H_2)(-G_5 H_3)$$

$$= G_2 G_3 G_5 H_2 H_3$$

4. Calculation of Δ and Δ_k.

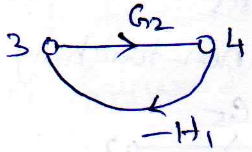
$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$$

$$= 1 - (-G_2 H_1 - G_2 G_3 H_2 - G_5 H_3) + (G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3)$$

$$= 1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3$$

$\Delta_1 = 1$, Since there is no part of graph which is not touching with first forward path.

The part of the graph which is not touching with second forward path is



$$\Delta_2 = 1 - P_{11}$$

$$= 1 - (-G_2 H_1)$$

$$\Delta_2 = 1 + G_2 H_1$$

5. Transfer function,

By Mason's gain formula, the transfer function

$$T = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

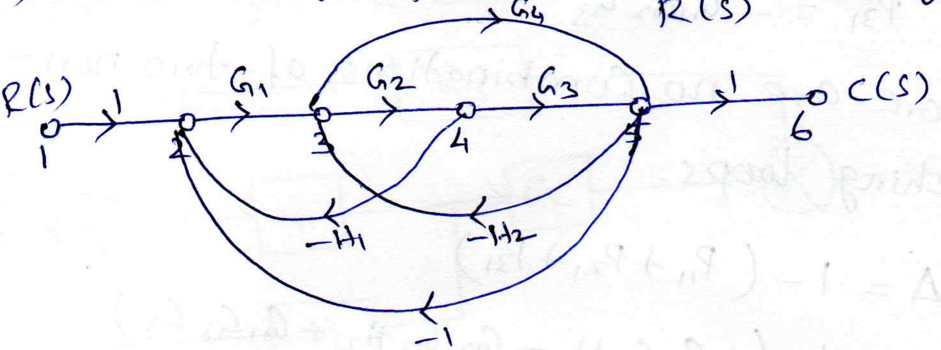
$$= \frac{1}{\Delta}(P_1 \Delta_1 + P_2 \Delta_2) \quad (\because K = 2)$$

$$\therefore T = \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 (1 + G_2 H_1)}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

$$= \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 + G_2 G_4 G_5 G_6 H_1}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

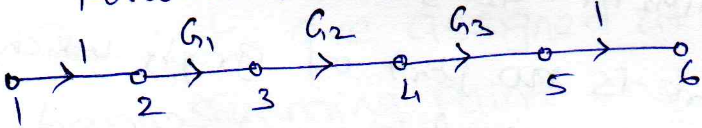
$$T = \frac{G_2 G_4 G_5 [G_1 G_3 + (G_6 / G_2) + G_6 H_1]}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

3) Find transfer function $\frac{C(s)}{R(s)}$ (Home work)

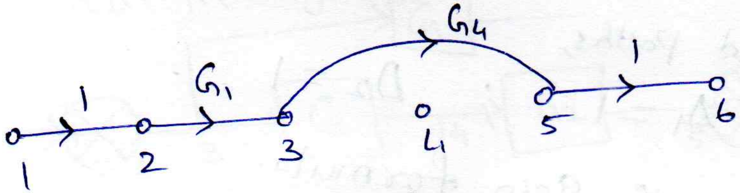


Soln:

Forward path K = 2

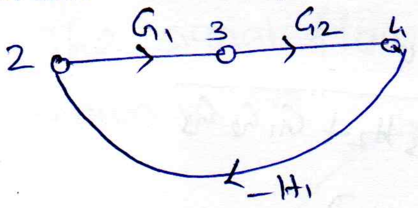


$$P_1 = G_1 G_2 G_3$$

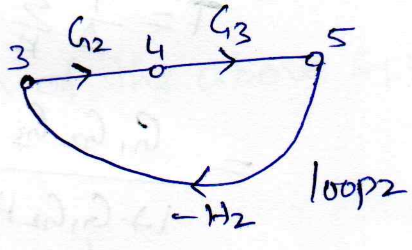


$$P_2 = G_1 G_4$$

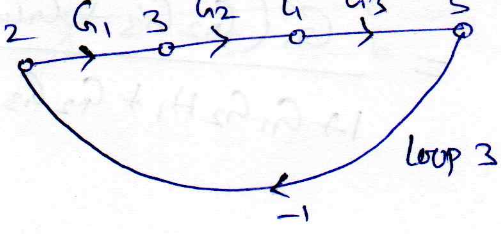
Individual loop:



loop 1



loop2



loop 3

$P_{11} = -G_1G_2H_1$ $\quad$ $P_{21} = -G_2G_3H_2$

$P_{31} = -G_1G_2G_3$

There are no combinations of two non-touching loops.

$$\therefore \Delta = 1-(P_{11}+P_{21}+P_{31})$$

$$= 1-(-G_1G_2H_1 - G_2G_3H_2 - G_1G_2G_3)$$

$$= 1+G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

Since, there is no part of graph which is not touching with first and second forward paths,

$$\Delta_1 = 1 \quad ; \quad \Delta_2 = 1$$

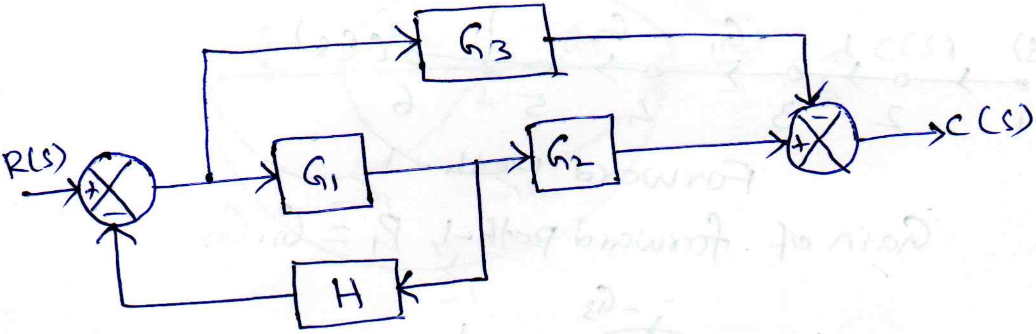
Mason's gain formula.

$$T = \frac{1}{\Delta}\sum_K P_K \Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1+G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$$

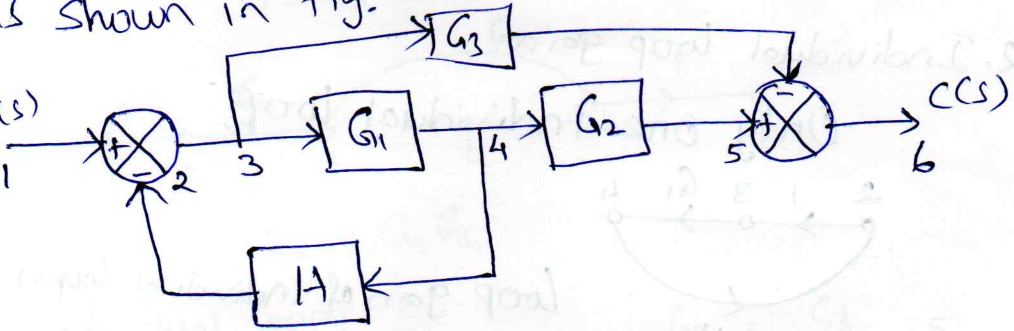
$$= \frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1+G_2G_3H_2+G_1G_2G_3}$$

4) Convert the given block diagram to signal flow graph and determine C(S) / R(S)

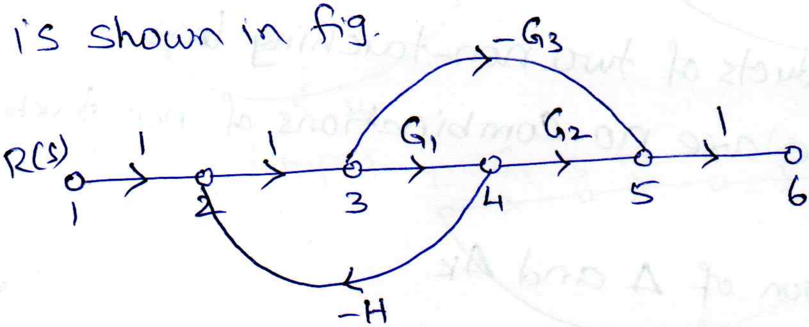


Soln.:

The nodes are assigned at input, output, at every summing point and branch point as shown in fig.

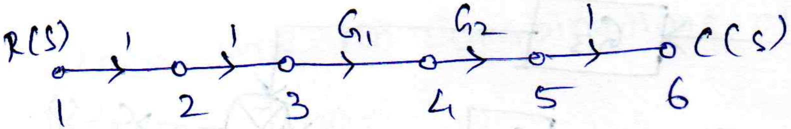


The signal flow graph of the above system is shown in fig.



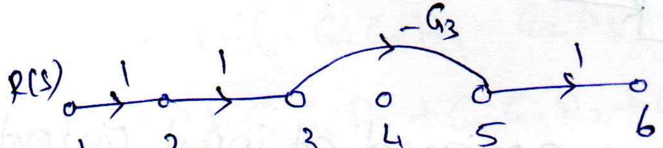
1. Forward path gains:

Number of forward path, $k = 2$



Forward Path - 1

Gain of forward path-1, $P_1 = G_1 G_2$

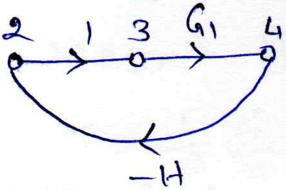


Forward path - 2

Gain of forward path-2, $P_2 = -G_3$

2. Individual loop gain:

Only one individual loop,



Loop 1.

Loop gain of individual loop-1

$P_{11} = -G_1 H$

3. Gain Products of two non-touching loops:

There are no combinations of non-touching loops.

4. Calculation of Δ and Δ_k.

$$\Delta = 1 - [P_{11}]$$

$$= 1 + G_1 H$$

Since, there are no part of the graph which is non touching with forward path-1 and forward Path-2,

$$\Delta_1 = 1$$

$$\Delta_2 = 1$$

5. Transfer function:

By Mason's gain formula,

$$T(S) = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$= \frac{1}{\Delta}\left[P_1 \Delta_1 + P_2 \Delta_2\right]$$

$$T(S) = \frac{G_1 G_2 - G_3}{1 + G_1 H}$$

The End

Maximum overshoot / peak overshoot (M_p):

→ It is the difference between peak value of time response and steady state value.

$$\% \; M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

where
$c(\infty)$ → final value of $c(t)$
$c(t_p)$ → Max value of $c(t)$

→ $$M_p = \frac{c(t_p) - 1}{1} \times 100$$

$\because c(\infty) = 1$

$\cancel{M_p} = \cancel{c(t_p)} =$ $\boxed{M_p = c(t_p) - 1}$

$t = t_p$

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left[\sin(\omega_d t + \theta) \right]$$

$$c(t_p) = 1 - \frac{e^{-\xi \omega_n \pi/\omega_d}}{\sqrt{1-\xi^2}} \times \sin(\pi/\cancel{\omega_d} \times \cancel{\omega_d} + \theta)$$

$$c(t_p) = 1 - \frac{e^{-\xi \omega_n \pi/\omega_d}}{\sqrt{1-\xi^2}} \cdot \sin(\pi + \theta)$$

$$c(t_p) = 1 + \frac{e^{-\xi\omega_n \pi/\omega_d}}{\sqrt{1-\xi^2}} \sin\theta$$

$$c(t_p) = 1 + \frac{e^{-\xi\omega_n\pi/\omega_d}}{\cancel{\sqrt{1-\xi^2}}} \times \cancel{\sqrt{1-\xi^2}}$$

$$c(t_p) = 1 + e^{-\xi \cancel{\omega_n} \pi / \cancel{\omega_n}\sqrt{1-\xi^2}}$$

$$c(t_p) = 1 + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

$$\% M_p = (\cancel{1} + e^{-\xi\pi/\sqrt{1-\xi^2}} - \cancel{1}) \times 100$$

Setteling time(t_s):

The setteling time 't_s' is defind as the time requared for the response to reach final value with in the specifed tolerance band i.e., 2% or 5%.

for 2% $t_s = \frac{4}{\xi\omega_n}$

for 5% $t_s = \frac{3}{\xi\omega_n}$

Steady state error:

Steady state error is defined as the difference between input r(t) and output c(t) of the system as time goes to infinity (i.e., when response has reached to steady state).

→ It is denoted as $\boxed{e_{ss}}$

→ Steady state error can be calculated by using final value theorem.

~~final value theorem~~ ~~e_{ss} or $e_\infty = \lim_{t\to\infty} e(t)$~~

~~laplace transform~~ ~~$e_{ss} = \lim_{t\to\infty} s E(S)$~~

$$\cancel{\boxed{e_{ss} \text{ q } e_\infty = \lim_{t\to\infty} e(t)}}$$

~~Apply laplace transform~~

$$\cancel{\boxed{e_{ss} = \lim_{t\to\infty} s E(S)}}$$

→ Steady state error can be calculated by using final value theorem.

$$\cancel{f}(t) = F(s)$$

Final value theorem for laplace Transform.

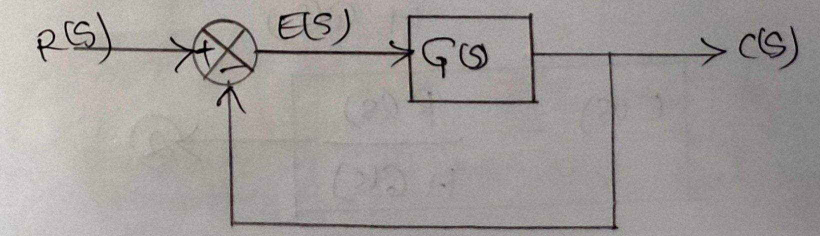
$$\boxed{\begin{array}{l} L[f(t)] = F(S), \text{ then} \\ \underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt} s F(S) \end{array}}$$

for steady state error e_{ss} is given by

$$e_{ss} = \underset{t\to\infty}{Lt} e(t)$$

$$\boxed{e_{ss} = \underset{s\to 0}{Lt} s E(S)} \rightarrow ①$$

To calcalate E(S), consider closed loop feedbook system with unity feedback.



$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

where $H(S) = 1$

$$C(S) = \frac{R(S)\,G(S)}{1+G(S)}$$

Error $E(S) = R(S) - C(S)$

$$E(S) = R(S) - \frac{R(S)G(S)}{1+G(S)}$$

$$= \frac{R(S)(1+G(S)) - R(S)\,G(S)}{1+G(S)}$$

$$= \frac{R(S) + \cancel{R(S)G(S)} - \cancel{R(S)G(S)}}{1+G(S)}$$

$$\boxed{E(S) = \frac{R(S)}{1+G(S)}} \rightarrow ②$$

Substitute eq ② in eq ①

$$\boxed{e_{ss} = \lim_{S\to 0} \frac{S\,R(S)}{1+G(S)}} \rightarrow ③$$

Causes of steady state error

- Nature of inputs
- Types of system
- Non - linearity of system components

Static error constants;

- When input is applied to control system, the steady state error may zero, constant or infinity.
- Value of steady state error depends on type of system and input signal.

→ There are three types of static error constants.

- ↳ positional error constant
- ↳ velocity error constant
- ↳ Acceleration error constant

Unit step Input:

Consider unit step input $r(t)$

$$r(t) = u(t) = 1$$

$$L\{r(t)\} = R(s) = \frac{1}{s}$$

Substitute $R(s)$ in eq ③

$$e_{ss} = \underset{s\to 0}{Lt} \frac{\cancel{s} \times \frac{1}{\cancel{s}}}{1+G(s)}$$

$$\boxed{e_{ss} = \frac{1}{1+\underset{s\to 0}{Lt} G(s)}}$$

$$e_{ss} = \frac{1}{1+K_p}$$

$$\boxed{K_p = \underset{s\to 0}{Lt} G(s)}$$

where

K_p – position error constant.

Consider poles and zero equation

$$G(s) = \frac{K.(s+z_1)(s+z_2)+(s+z_3)+\cdots}{s^N(s+p_1)(s+p_2)+(s+p_3)+\cdots}$$

Type 0 system:

$$K_p = \underset{s\to 0}{Lt} G(s)$$

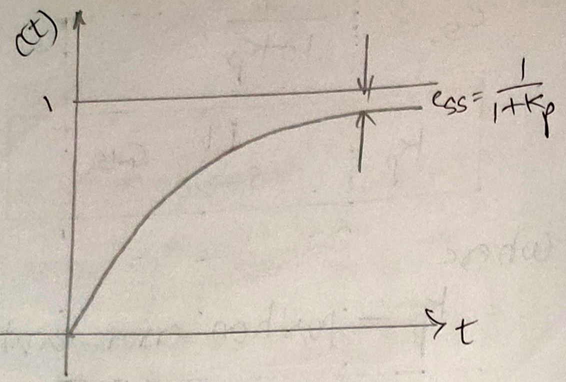
$$= \underset{s\to 0}{Lt} K \frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

$$= K \frac{z_1 z_2 z_3}{p_1 p_2 p_3}$$

$$\boxed{K_p = K}$$

In this case, the steady state error is constant

$$e_{ss} = \underset{s\to 0}{Lt} \frac{1}{1+K_p}$$



Type I system:

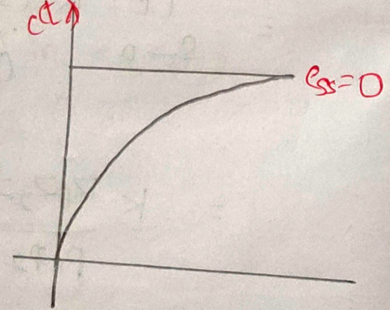
$$G(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)}{s(s+p_1)(s+p_2)(s+p_3)}$$

$$K_p = \lim_{s\to 0} \frac{K z_1 z_2 z_3}{0(p_1 p_2 p_3)}$$

$$\boxed{K_p = \infty}$$

$$e_{ss} = \frac{1}{1+\underset{\to\infty}{K_p}}$$

$$\boxed{e_{ss} = 0}$$



$K_p=\infty$ and $e_{ss}=0$
~~A~~ step input 4 for
higher order systems

N	K_p	e_{ss}
0	K	$\frac{1}{1+K}$
1	∞	0
2	∞	0

Ramp Input:

Ramp input

$r(t) = t$

$L\{r(t)\} = R(s) = \frac{1}{s^2}$

Consider, steady state error equation

$$e_{ss} = \lim_{s\to 0} \frac{R(s)}{1+G(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{\cancel{s}\cdot\frac{1}{s^{\cancel{2}}}}{1+G(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{s(1+G(s))}$$

$$e_{ss} = \underset{s \to 0}{Lt} \frac{1}{s + sG(s)}$$

Now,

$$e_{ss} = \frac{1}{0 + \underset{s \to 0}{Lt} sG(s)}$$

$$\boxed{\therefore e_{ss} = \frac{1}{\underset{s \to 0}{Lt} sG(s)}}$$

$$\boxed{e_{ss} = \frac{1}{K_v}}$$

$$K_v = \underset{s \to 0}{Lt} sG(s)$$

where

K_v → called as velocity error constant

Type 0 system:

$$G(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

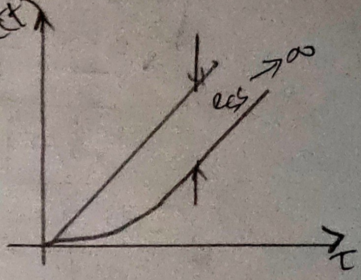
$$K_v = \underset{s \to 0}{Lt} sG(s)$$

$$K_v = \underset{s \to 0}{Lt} s \frac{K(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

If s=0 $\boxed{K_v = 0}$

$$e_{ss} = \frac{1}{K_v}$$

$$\boxed{e_{ss} = \infty}$$



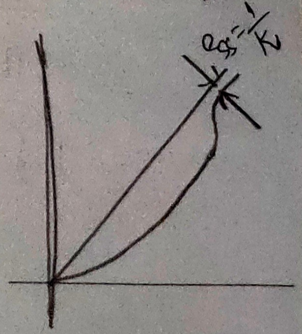
Type 1 system:

$$K_v = \underset{s \to 0}{Lt} \cancel{s} \times \frac{K(s+z_1)(s+z_2)(s+z_3)}{\cancel{s}(s+p_3)(s+p_2)(s+p_3)}$$

If s=0

then $\boxed{K_v = K}$

$$\boxed{e_{ss} = \frac{1}{K_v} = \frac{1}{K}}$$



~~Stad~~ Steady state for type 1 system ~~so~~ for ramp input is constant.

Type 2 system:

$$K_v = \lim_{s\to 0} s\,\frac{K(s+z_1)(s+3)\cdots}{s^2(s+p_1)(s+p_2)\cdots}$$

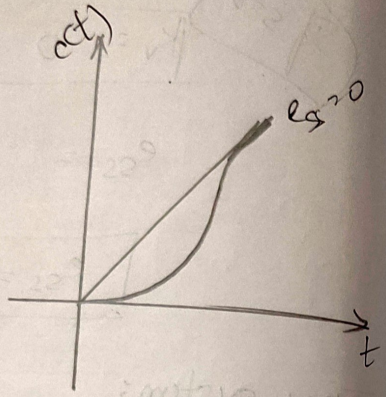
If s=0

$$K_v = \frac{1}{0}$$

$$\boxed{K_v = \infty}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty}$$

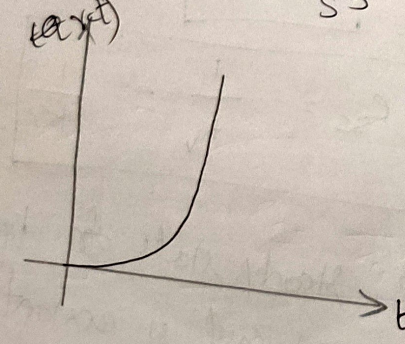
$$\boxed{e_{ss} = 0}$$



Parabolic Input:

$r(t) = t^2$

$L\{r(t)\} = R(s) = \frac{1}{s^3}$



$$e_{ss} = \lim_{s\to 0} \frac{s \times \frac{1}{s^3}}{1+G(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{s^2(1+G(s))}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)}$$

If s=0, then

$$e_{ss} = \frac{1}{0+\lim_{s\to 0} s^2G(s)}$$

$$e_{ss} = \frac{1}{\lim_{s\to 0} s^2G(s)}$$

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

$$\boxed{K_a = \lim_{s\to 0} s^2G(s)}$$

Where

K_a - called as acceleration error constant.

Type 'o' system:

$$K_a = \lim_{s \to 0} s^2 \frac{K(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

if $s=0$

$$\boxed{K_a = 0}$$

$$e_{ss} = \frac{1}{0}$$

$$\boxed{e_{ss} = \infty}$$

Type 1 system:

$$K_a = \lim_{s \to 0} s^{\cancel{2}} \frac{K(s+z_1)(s+z_2)}{\cancel{s}(s+p_1)(s+p_2)}$$

$$K_a = \lim_{s \to 0} 0$$

$$\boxed{K_a = 0}$$

$$\boxed{e_{ss} = \frac{1}{K_a} = \frac{1}{0} \to \infty}$$

Type 2 system:

$$K_a = \lim_{s \to 0} \frac{\cancel{s^2} K(s+z_1)(s+z_2)}{\cancel{s^2}(s+p_1)(s+p_2)}$$

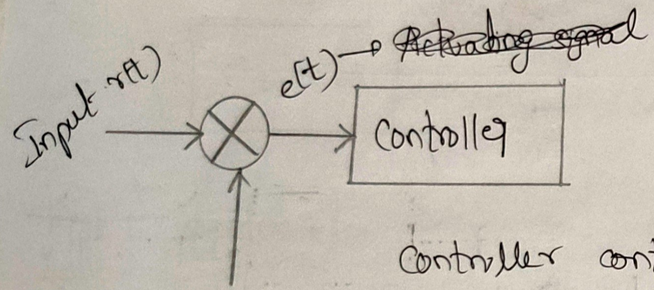
$$\boxed{K_a = K}$$

$$\boxed{e_{ss} = \frac{1}{K}}$$

Controllers:

- A controller is the most important component or device & algorithm of the control system.

- Controller makes error to zero & to lowest values.



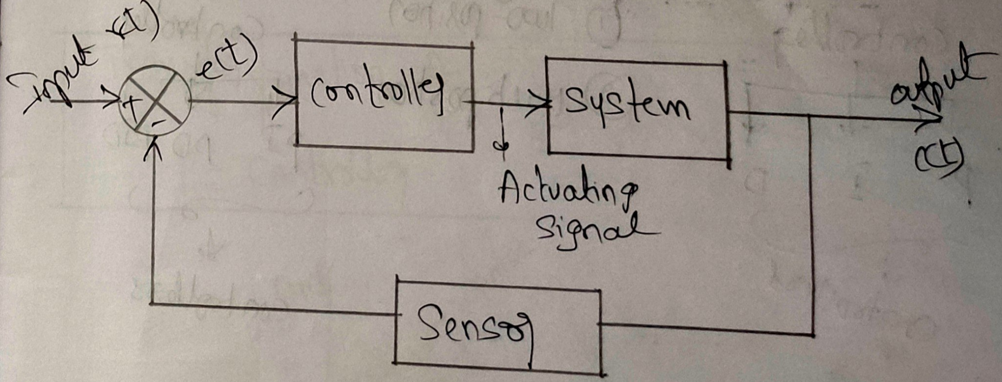
Controller controls the system characteristic & system behavior.

- Controller receives error signal based on this, it generates a control action to make error to zero or lowest value

- With the help of controller accurate output can be obtained.

Uses of controllers:

- Decreasing steady state error
- If e_{ss} decreases, the stability also improves.
- helps to speed up the slow response of overdamped system.
- reducing noise signals.



Types of Controllers:

There are two main types of controllers.
- Discontinuous controllers.
- continuous controllers

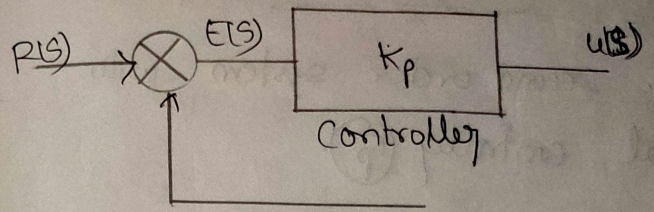
Controller types

→ Controllers are classified based on their control actions.

→ Controller
- Continuous controller
    - P
    - I
    - D
    - controllers
- Discontinuous controller
    - ① Two-position
    - ② Multiposition
- Composite controller
    - PI
    - PD
    - PID
    - controllers

proportional (p) controller:

→ The control signal is proportional to error signal.

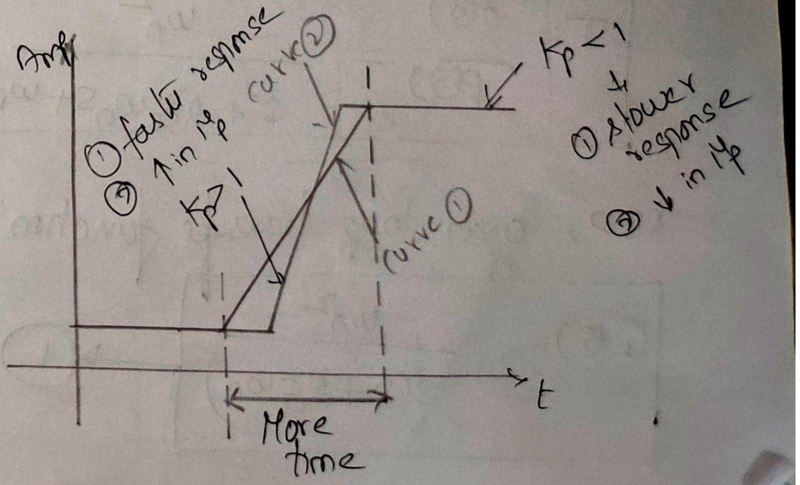


$U(S) \propto E(S)$

$$\boxed{U(S) = K_p E(S)}$$

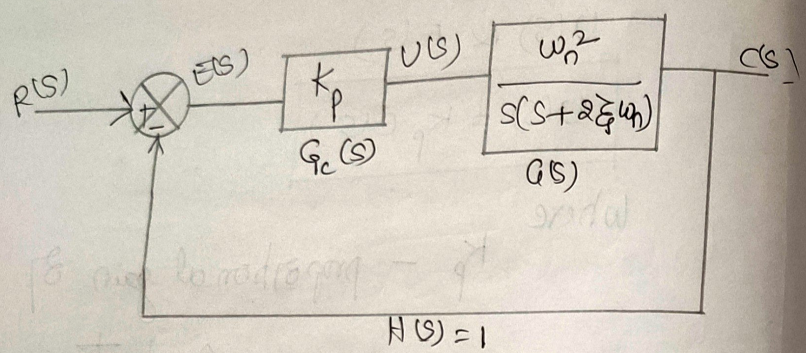
where K_p - proportional gain of constant

$$\boxed{\text{Transfer function of P- Controller} \quad \frac{U(S)}{E(S)} = K_p}$$



Effect of proportional (p) controller:

Consider, second order system with proportional controller (K_p).



Without controller:

The transfer function is

$$\boxed{T = \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}}$$

Now, open loop transfer function.

$$\boxed{G(S) = \frac{\omega_n^2}{S(S+2\xi\omega_n)}} \rightarrow ①$$

The system is type 1,

$$K_V = \lim_{S\to 0} S\,G(S)$$

$$K_V = \lim_{S\to 0} \cancel{S} \times \frac{\omega_n^2}{\cancel{S}(S+2\xi\omega_n)}$$

$$K_V = \frac{\omega_n^2}{0+2\xi\omega_n}$$

$$K_V = \frac{\omega_n^{\cancel{2}}}{2\xi\cancel{\omega_n}}$$

$$\boxed{K_V = \frac{\omega_n}{2\xi}} \rightarrow ②$$

$$e_{ss} = \frac{1}{K_V}$$

$$e_{ss} = \frac{1}{\frac{\omega_n}{2\xi}}$$

$$\boxed{e_{ss} = \frac{2\xi}{\omega_n}} \rightarrow ③$$

With controller

open loop transfer function $= \frac{K_p \omega_n^2}{s(s+2\xi\omega_n)}$

Here also type 1 system

$$K_v = \lim_{s \to 0} s \times G_c(s) G_p(s)$$

$$= \lim_{s \to 0} \cancel{s} \times \frac{K_p \omega_n^2}{\cancel{s}(s+2\xi\omega_n^2)}$$

$$K_v = \frac{K_p \omega_n^{\cancel{2}}}{2\xi\cancel{\omega_n}}$$

$$\boxed{K_v = \frac{K_p \omega_n}{2\xi}}$$

$$e_{ss} = \frac{1}{K_v}$$

$$e_{ss} = \frac{1}{\frac{K_p\omega_n}{2\xi}}$$

$$\boxed{e_{ss} = \frac{2\xi}{K_p \omega_n}}$$

∴ Steady state error depends on ⓚK_p

$$\boxed{e_{ss} \propto \frac{1}{K_p}}$$

$$\boxed{\text{If } K_p \uparrow \downarrow \text{ Then } e_{ss} \downarrow \uparrow}$$

$$\boxed{s^2 + 2\xi\omega_n s + \omega_n^2} = \boxed{s^2 + 2\xi\omega_n s + K_p\omega_n^2}$$

$$(\omega_n')^2 = K_p \omega_n^2$$

$$\boxed{\omega_n' = \sqrt{K_p}\,\omega_n}$$

$$2\xi\omega_n' s = 1$$

$$\boxed{2\xi\sqrt{K_p}\,\omega_n = 2\xi\omega_n s}$$

↓ compare

$$2\xi\sqrt{K_p}\,\omega_n = 1$$

with K_p, the value of $\xi \downarrow$ ←

$$\boxed{\xi = \frac{1}{2\sqrt{K_p}\,\omega_n}}$$

$$2\xi\omega_n s = 1$$

$$\boxed{\xi = \frac{1}{2\omega_n}}$$

↓ value constant

The effects of proportional controller are

↳ ① type and order of the system donot changed.

↳ ② damping ratio decreases hence overshoot increases.

↳ ③ T.F remains same.

↳ ④ steady state error ~~de~~ reduces which indicates that accuracy is improved.

→ P-controller cannot eliminate, complete e_{ss}.

UNIT-V

State space Analysis of continuous systems

Introduction:

→ The analyse MIMO systems, state variable technique used

state space technique ↳ which reduces complexity.

→ This technique, determines internal behavior of the system.

Conventional Methods

↳ Bode plot and Nyquist plot are frequency domain requires laplace Transform for continuous time systems and z-transform for discrete time systems.

→ But for both continuous and discrete-time systems vector matrix form of state-space representation ~~reduces~~ greatly ~~simplify~~ simplifies system representation and gives accurately.

concept of state, state variables and state model:

→ In state space analysis time variant systems can be defined.

→ Individual variables can be defined in the MIMO systems.

Velocity $\frac{dy}{dt}$

State:

→ It represents every smallest part information of the system in order to predict response.

→ Basically, a state separates the future ie., the response of the system from past.

(or)

→ It gives condition of system at any instant of time.

State variable:

The state variable are a set of variables that completely describe the state or condition of system at any instant of time.

State vector:

→ This is a vector consisting of ⓝ number of state variables that completely determine the behavior of the system.

State space:

→ The state space is an n-dimensional space whose coordinate axis are ⓝ number of state variables that completely determine the behavior of system.

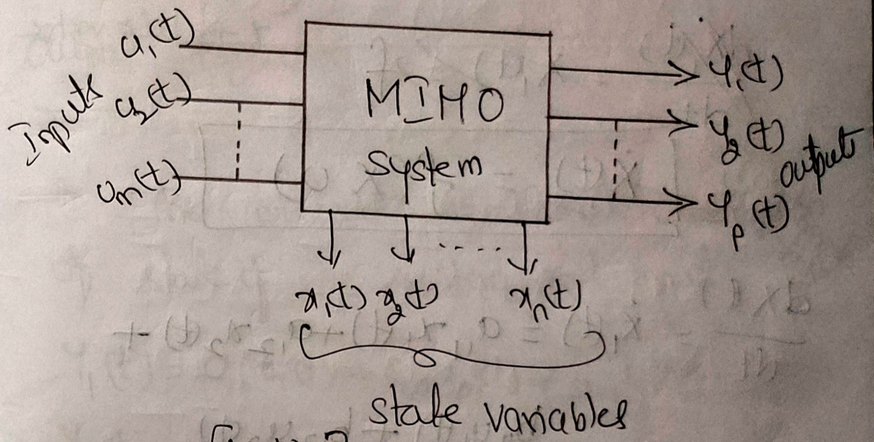
Ex:

If $x_1, x_2 \cdots x_n$ are n-state variables, then its state vector (X) is written by an $n \times 1$ matrix as follows

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ where $x_1, x_2 \cdots x_n$ all are functions of ⓣ

State Model:

Consider Multiple Input Multiple output n^{th} order system as shown in fig.



$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{bmatrix}$$

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_p(t) \end{bmatrix}$$

where m — no. of inputs

P — no of output

→ All are column vectors having orders $m \times 1$, $n \times 1$ and $p \times 1$.

→ For such system, the state variable representation can be arranged in the form of ⓝ first order differential equations

$$\cancel{\frac{dX_1(t)}{dt} = X_1(t) \Rightarrow f}$$

$x \rightarrow$ state vector

$$\boxed{\dot{X}(t) = f(X, U)}$$

derivation of state vector

$$\frac{dX_1(t)}{dt} = \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_{11}u_1(t) + b_{12}u_2(t) \quad \hookrightarrow ①$$

$$\frac{dX_2(t)}{dt} = \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_{21}u_1(t) + b_{22}u_2(t) \quad \hookrightarrow ②$$

Equation ① & ② can be represented in matrix form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \overset{\rightarrow 2\times 2}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \overset{\rightarrow 2\times 1}{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}} + \overset{\rightarrow 2\times 2}{\begin{bmatrix} b_{11} & b_{21} \\ b_{21} & b_{22} \end{bmatrix}} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

∴ Generalized form

$$\boxed{\dot{X}(t) = AX(t) + BU(t)}$$

→ The above equation is a state equation.

Output Equation:

→ The output equation combination of state of the system and input.

$$Y = g(X, U)$$

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) + d_{11}u_1(t) + d_{12}u_2(t) \quad ③$$

$$y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + d_{21}u_1(t) + d_{22}u_2(t) \quad ④$$

The c and d are coefficients and are constants.

→ Write equations ③ & ④ in matrix form.

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

→ Write in general form

$$\boxed{Y(t) = CX(t) + DU(t)}$$

The two equations combinedly form the state model of the linear system

$$\therefore \boxed{\begin{aligned} \dot{X}(t) &= A\,X(t) + B\,U(t) \\ Y(t) &= C\,X(t) + D\,U(t) \end{aligned}}$$

→ for linear time invariant system

where A, B, C, D are constant matrices

$$\dot{X}(t) = A(t)\,X(t) + B(t)\,U(t)$$

$$Y(t) = C(t)\,X(t) + D(t)\,U(t)$$

For linear time variant system.

where A, B, C, D are time dependent matrices

Order of matrices:

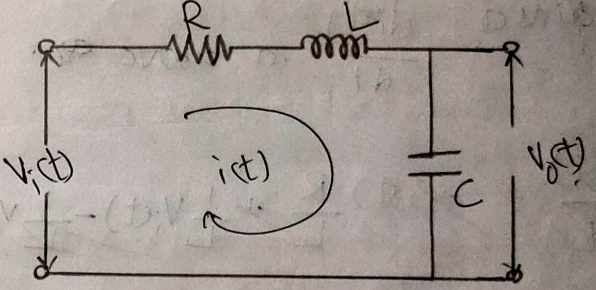
A → $n \times n$ called Evolution matrix

B → $n \times m$ called control matrix

C → $p \times n$ called observation matrix

D → $p \times m$ called Direct Transmission matrix

problem obtain the state model of the given electrical network in the standard form. Given at $t = t_0$, $i(t) = i(t_0)$ and $V_0(t) = V_0(t_0)$.



sol: There are two energy storage elements Ⓛ and Ⓒ so two two state variables are required.

$$X_1(t) = i(t) \text{ and}$$

$$X_2(t) = V_0(t)$$

$$X(t_0) = \begin{bmatrix} i(t_0) \\ V_0(t_0) \end{bmatrix}$$

Now, write differential equations for the given variables by using KCL and KVL.

$$V_i(t) = i(t)R + L\frac{di(t)}{dt} + \underbrace{\frac{1}{C}\int i(t)\,dt}_{V_0(t)}$$

↳ ①

state equation 4 output equation.

obtained $\frac{di(t)}{dt}$ in above eq:

$$\frac{di(t)}{\cancel{dt}} = \cancel{-i(t)}\frac{R}{\cancel{L}} + \frac{1}{L}V_i(t) - \cancel{\frac{1}{L}V_0(t)}$$

$$\frac{di(t)}{dt} = -i(t)\frac{R}{L} + \frac{1}{L}V_i(t) - \frac{1}{LC}\int i(t)\,dt$$

$$\boxed{\frac{dq}{dt} = i(t)}$$

$$\frac{di(t)}{dt} = -i(t)\frac{R}{L} + \frac{1}{L}V_i(t) - \frac{q}{LC} \qquad \boxed{\frac{dq}{dt} = i}$$

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ q \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} V(t)$$

$$\dot{X} = AX + BU$$

output $V_0(t) = X_2(t)$

system can be represented

↳ ① T.F model

↳ ② Differential equation model

↳ ③ Electrical n/w

↳ ④ SFG 4 Block diagram

State-space representation:

→ The state-space model of a system can be obtained in three different ways when differential equations or transfer function is given.

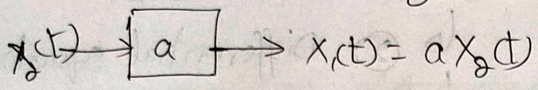
→ These different ways of representing a system in state space model.

Types of representation

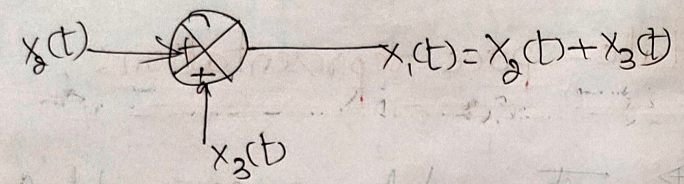
- physical variable form
- phase variable form
- canonical form

→ state diagram consists ③ elements

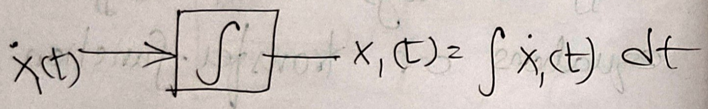
↳ scalars → it is like amplifiers having gain



↳ Adders → are summing points



↳ Integrators :-



↳ which integrates, the differentiate of state variable to obtain required state variable.

Standard state model representation

consider standard state model

$$\dot{X}(t) = A X(t) + B U(t) \text{ and}$$

$$Y(t) = C X(t) + D U(t)$$

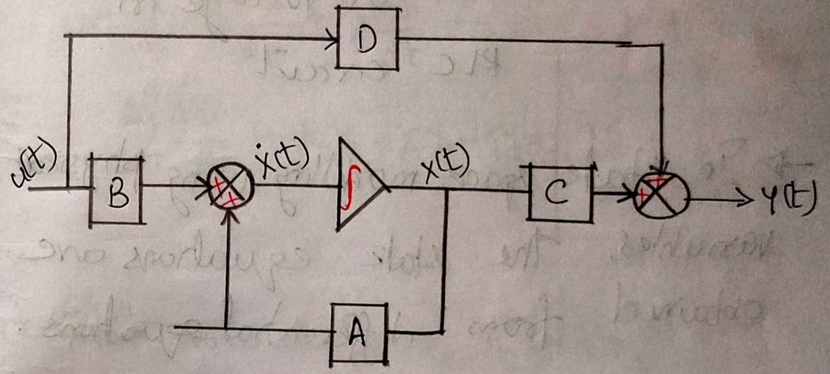


fig: Block diagram of state model

Derivation of state models from schematic models:

In state space analysis, the choice of state variables are random.

→ One of the choice of state variables is physical variables.

→ Ex:

① Mechanical systems → displacement, velocity, acceleration

② Electrical systems → physical variables are

↓ current or voltage in RLC circuit.

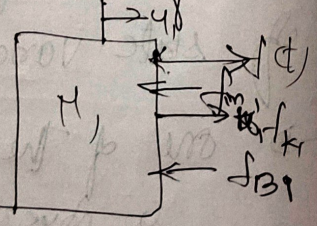
→ In state space modelling using physical variables, the state equations are obtained from differential equations.

↳ it is the basic model.

problem

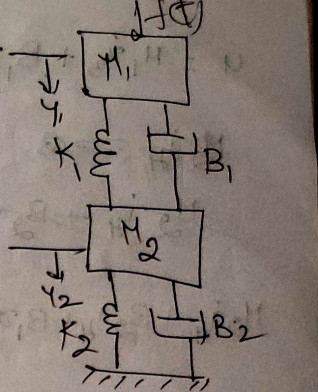
construct the state model of mechanical model as shown.

Sol: free body diagram for M_1



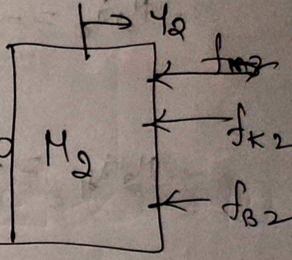
$$M_1\frac{d^2y_1}{dt}+B\frac{dy_1}{dt}-B_1\frac{dy_2}{dt}+K_1y_1-K_1y_2=f(t)$$

↳ ①



$$M_2\frac{d^2y_2}{dt}+B_2\frac{dy_2}{dt}+B\frac{dy_1}{dt}-B_1\frac{dy_1}{dt}+K_2y_2+Ky_2-Ky_1=0$$

↳ ②



choose state variables x_1, x_2, x_3 and x_4.

input $f(t)=u$.

relate state variables to physical variables

$\boxed{x_1=y_1}$; $\boxed{x_2=y_2}$; $\boxed{x_3=\frac{dy_1}{dt}}$; $\boxed{x_4=\frac{d^2y_2}{dt}}$

$\boxed{\dot{x}_3=\frac{d^2y_1}{dt^2}}$ and $f(t)=u$

$$u = M_1 \dot{x}_3 + B_1 x_3 - B_1 x_3 + K_1 x_2 + K_2 x_2 - K_1 x_1 = 0$$

~~$M_2 \dot{x}_4 = -B$~~

~~$M_2 \dot{x}_4 + B_2 x_4 + B_1 x_4 + B_1 x_3 - K_2 x_2 - K_1 x_2 + K_1 x_1 = 0$~~

$$M_2 \dot{x}_4 + B_2 x_4 + B_1 x_4 - B_1 x_3 + K_1 x_2 + K_2 x_2 - K_1 x_1 = 0$$

$$\therefore M_2 \dot{x}_4 = -B_2 x_4 - B_1 x_4 + B_1 x_3 - K_2 x_2 - K_1 x_2 + K_1 x_1$$

$$M_2 \dot{x}_4 = -(B_2 - B_1) x_4 + B_1 x_3 - (K_2 + K_1) x_2 + K_1 x_1$$

$$\therefore \dot{x}_4 = \frac{K_1}{M_2} x - \frac{(K_1 + K_2)}{M_2} x_2 + \frac{B_1}{M_2} x_3 - \frac{(B_1 + B_2)}{M_2} x_4$$

state variable $x_1 = y_1$

differenting $x_1 = y_1$ w.r.ⓣ

$$\frac{dx_1}{dt} = \dot{x}_1 \text{ and } \frac{dy_1}{dt} = x_3;$$

State variable $\boxed{\dot{x}_1 = x_3}$

$$x_2 = y_2$$

$$\frac{dx_2}{dt} = \frac{dy_2}{dt}$$

$$\frac{dx_2}{dt} = \dot{x}_2 \text{ and } \frac{dy_2}{dt} = x_4;$$

$$\boxed{\therefore \dot{x}_2 = x_4}$$

$$\dot{x}_1 = x_3 \text{ and } \dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{K_1}{M_1} x_1 + \frac{K_1}{M_1} x_2 - \frac{B_1}{M_1} x_3 + \frac{1}{M_1} u$$

$$\dot{x}_4 = \frac{K_1}{M_2} x_1 - \frac{(K_1 + K_2)}{M_2} x_2 + \frac{B_1}{M_2} x_3 - \frac{B_1 + B_2}{M_2} x_4$$

On arranging state equations in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \left[\right.$$

$$T(S) = \frac{s^2+3s+3}{s^3+2s^2+3s+1}$$

$$T(S) = \frac{Y(S)}{\cancel{2}U(S)} = \frac{X(S)}{U(S)} \times \frac{Y(S)}{X(S)}$$

$$= \left[\frac{1}{s^3+2s^2+3s+1}\right] \times \left[s^2+3s+3\right]$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^3+2s^2+3s+1} \rightarrow \text{state equation}$$
↳ ①

$$\frac{Y(s)}{X(s)} = s^2+3s+3 \rightarrow \text{output eq.}$$
↳ ②

from eq ①

$$U(S) = S^3X(s) + 2S^2X(S) + 3SX(s) + X \quad ③$$

Taking inverse L ↳ order 3

$$u(t) = \dddot{x} + 2\ddot{x} + 3\dot{x} + x$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} = \dot{x}_1 \\ x_3 &= \ddot{x} = \dot{x}_2 \\ \dot{x}_3 &= \dddot{x} \end{aligned}\right\} \rightarrow$$

state equation

$$\dot{X} = AX + BU$$

$$u(t) = \dot{x}_3 + 2x_3 + 3x_2 + x_1$$

$$\boxed{\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + u(t)}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

output Equation

$$Y(s) = s^2 X(s) + 3sX(s) + 3X(s)$$

By taking inverse laplace transform

$$Y(t) = \dddot{X} + 3X + 3X$$

$$Y(t) = x_3 + 3x_2 + 3x_1$$

Re arrange

$$Y(t) = 3x_1 + 3x_2 + x_3$$

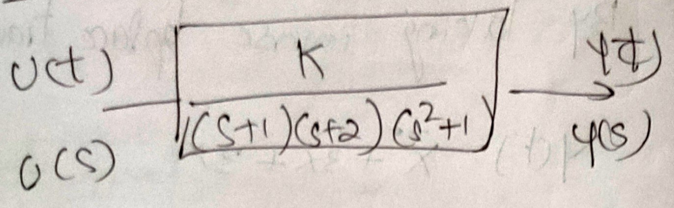
output eq.

$$Y = CX + DU$$

$$y(t) = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0][U]$$

②

Transfer function



$$G(S) = \frac{Y(S)}{U(S)} = \frac{K}{(S+1)(S+2)(S^2+1)}$$

$$Y(S)(S+1)(S+2)(S^2+1) = KU(S).$$

$$(S^4 + 3S^3 + 3S^2 + 3S + 2)\,Y(S) = KU(S)$$

$$\downarrow$$

Taking Inverse Laplace Transform

$$\frac{d^4y(t)}{dt^4} + \frac{3\,d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = Ku(t)$$

$$\ddddot{y}(t) + 3\dddot{y}(t) + 3\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = Ku(t)$$
$\to$ ①

Choose state variables

let $y(t) = x_1$

$$\dot{y}(t) = \dot{x}_1 = x_2$$
$$\ddot{y}(t) = \ddot{x}_1 = \dot{x}_2 = x_3$$
$$\dddot{y}(t) = \dddot{x}_1 = \ddot{x}_2 = \dot{x}_3 = x_4$$
$$\ddddot{y}(t) = \ddddot{x}_1 = \dddot{x}_2 = \ddot{x}_3 = \dot{x}_4$$

substitute state variables in eq ①

$$\boxed{\dot{x}_4 + 3x_4 + 3x_3 + 3x_2 + 2x_1 = KU}$$
→ ②

$$\dot{x}_3 = x_4$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_1 = x_2$$
→ state equations.

eq ② can be written as

$$\dot{x}_4 = -2x_1 - 3x_2 - 3x_3 - 3x_4 + KU$$

State model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -3 & -3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} u$$

→ If a matrix is of the form

output equation $y(t) = x_1$ by matrix (A)

Then it is called bush or

~~Ptd~~= $y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ companion form

Canonical forms;

→ canonical forms are the standard forms of state model

→ There are several canonical forms

companion forms
- ① phase variable canonical form
- ② controllable canonical form
- ③ observable canonical form

modal forms
- ④ digonal canonical form
- ⑤ Jordan canonical form

Controllable canonical form

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Rewrite above equation in equal order in Nr & Dr

$$\frac{Y(s)}{U(s)} = \frac{0s^2 + s + 3}{s^2 + 3s + 2}$$

||ly

compare with this y

$$\boxed{\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \dots + b_{n-1} s b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}}$$

get

a & b values

$b_0 = 0$, $b_1 = 1$ & $b_2 = 3$ → in output

$a_1 = 3$; $a_2 = 2$ → represented in state matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y = \begin{bmatrix} \cancel{b_2 - a_2 b_0 \quad b_1 - a_1 b_0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

$[b_3 \quad b_2 \quad b_1]$

$$Y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ → is important in pole placement approach.

Diagonal canonical form

Determine the state representation of continuous-time LTI system with system function $G(s) = \dfrac{(s+5)}{(s+1)(s+3)}$ in diagonal canonical form.

Sol:

$$G(s) = \frac{s+5}{(s+1)(s+3)}$$

Using partial-fraction expansion

$$\frac{s+5}{(s+1)(s+3)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)}$$

$$s+5 = A_1(s+3) + A_2(s+1)$$

Equating ~~[illegible]~~ coefficients (S) on both

$$1 = A_1 + A_2$$

Comparing constant ~~co~~-efficients

$$5 = 3A_1 + A_2$$

$$\boxed{A_1 = 2} \text{ and } \boxed{A_2 = -1}$$

$$\frac{s+5}{(s+1)(s+3)} = \frac{2}{(s+1)} ~~+~~ - \frac{1}{s+3}$$

$$P_1 = -1 \; ; \; P_2 = -3$$

$$C_0 = 0 \; ; \; C_1 = 2 \text{ and } C_2 = -1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

pration

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Jordan canonical form;

$$G(s) = \frac{1}{(s+3)^2(s+2)}$$

Partial fraction

$$\frac{1}{(s+3)^2(s+2)} = \frac{A_1}{(s+3)^2} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+2)}$$

$$1 = A_1(s+2)(+A_2(s+3)+A_3(s+3)^2$$

$$A_1 = -1, \; A_2 = -1 \text{ and } A_3 = 1$$

$$T(s) = - \frac{1}{(s+3)^2} - \frac{1}{(s+3)} + \frac{1}{s+2}$$

$$C_1 = 0 \; ; \; C_2 = -1, \; C_3 = -1$$

$$P_1 = -3 \text{ and } P_2 = -2$$

comparing the above eq with standard Jordan canonical form, the coefficient values for diagonal canonical form so

observable canonical form: [OCF]

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -a_2\\ 1 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}b_2-a_2b_0\\ b_1-a_1b_0\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + b_0u(t)$$

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problem Determine the state representation of a continuous-time LTI system with system function $G(s) = \frac{s+7}{(s+2)(s+3)}$ in observable canonical form.

sol:

$$G(s) = \frac{s+7}{(s+2)(s+3)}$$

$$\Downarrow$$

$$\boxed{s^2+3s+2s+6}$$

$$\Downarrow$$

Order $n=2$

$b_0=0; b_1=1$
$b_2=7$

$$\boxed{a_1=5;\ a_2=6}$$

s^2+5s+6

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -6\\ 1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}3\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonalization of state matrix:

The process of converting non-diagonal system matrix into diagonal system matrix.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 6 & 5 \end{bmatrix}}_{\text{non diag}} \Rightarrow \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}}_{\text{Non-diagnal matrix}}$$

State equation of the state model

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

Substitute $x(t) = MZ(t)$

$$M\dot{Z}(t) = AMZ(t) + BU(t)$$

$$\dot{Z}(t) = \frac{AMZ(t)}{M} + \frac{BU(t)}{M}$$

$$\boxed{\dot{Z}(t) = M^{-1}AMZ(t) + M^{-1}BU(t)}$$

M = Model matrix

$$M = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \text{ \& } C_2 = \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix}$$

Ex: obtain the diagonalization system matrix of a system which is described by the state equation.

$$\dot{X} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Diagonal system matrix $M^{-1}AM$

Finding Eigen values:

characteristics of equation of state model is

$$|S\underset{\downarrow}{I} - \overset{\rightarrow \text{system matrix } \underline{\underline{2\times2}}}{A}| = 0$$

Identity matrix

$$|SI - A| = \left| S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} S-3 & -4 \\ -2 & S-1 \end{bmatrix} \right|$$

$$= (S-3)(S-1) - 8$$

$$= S^2 - 4S - 5$$

$$= (S+1)(S-5)$$

$$= -1, -5$$

Finding eigen vectors:

For eigen values $S = -1$

$$[S_1 I - A] C_1 = 0$$

$$\left[-1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} \right] \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = 0$$

$$\begin{bmatrix} -4 & -4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = 0$$

$-4c_{11} - 4c_{21} = 0$

$-2c_{11} - 2c_{21} = 0$

let $c_{11} = 1,\ c_{21} = -1$

$$c_1 = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(1\times1)-(-1\times2)} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{8} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$M^{-1}AM =$

State Transition Matrix

→ For obtaining the solution of homogeneous and non-homogeneous state equations, it is necessary to determine state Transition Matrix (STM).

→ STM represented as $\boxed{\phi(t)}$ for homogeneous-type state equation

$$\boxed{\dot{x}(t) = Ax(t)}$$

Taking laplace Transform:

$$sX(s) - x(0) = AX(s)$$

$$[s\hat{I} - A]X(s) = x(0)$$

$$\boxed{X(s) = [s\hat{I} - A]^{-1} x(0)}$$

Taking Inverse laplace Transform

$$x(t) = L^{-1}\{[sI-A]^{-1} x(0)\}$$

$$\boxed{x(t) = L^{-1}\{[sI-A]^{-1}\} x(0)}$$

$$\boxed{\phi(t) = e^{At} = L^{-1}[sI-A]^{-1}}$$

$$\therefore L\{\phi(t)\} = L\{e^{At}\} = \phi(s) = [sI-A]^{-1}$$

$\phi(s) \rightarrow$ resolvent matrix.

$$\boxed{\phi(t) = e^{At}}$$

↳ is a state transition matrix.

properties:

① $\phi(0) = e^{A(0)} = I$

② $\phi(t) = e^{At} = e^{-(-At)^{-1}} = [\phi(-t)]^{-1}$

③ $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \phi(-t)$

④ $\phi(t_1+t_2) = e^{At_1} \cdot e^{At_2}$

$= \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

⑤ $[\phi(t)]^{K} = (e^{At})^{K} = \phi(Kt)$

⑥ $\phi(t_2-t_1)\phi(t_1-t_0) = \phi(t_2-t_0) = \phi(t_1-t_0)\phi(t_2-t_1)$

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controllable canonical form (ccf)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [b_3 \ b_2 \ b_1] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \underline{\underline{b_0 u(t)}}_{0}$$

$$y(t) = [b_3 \ b_2 \ b_1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observable canonical forms

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & -a_3\\ 1 & 0 & -a_2\\ 0 & 0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}b_3\\ b_2\\ b_1\end{bmatrix}u(t)$$

output equation

$b_0 = 0$

$$y(t) = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

Diagonal canonical form;

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-P_1 & 0 & 0\\ 0 & -P_2 & 0\\ 0 & 0 & -P_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u(t)$$

Output

$$y(t) = \begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + b_0 u(t)$$

Jordan canonical form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-P_1 & 1 & 0\\ 0 & -P_2 & 1\\ 0 & 0 & -P_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}c_1 & c_2 \cdots c_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + b_0 u(t)$$

$$\frac{Y(S)}{U(S)} = \frac{S+3}{S^2+3S+2}$$

$$Y(S)S^2 + 3S\,Y(S) + 2Y(S) = U(S)S + U(S)3$$

$$\frac{Y(S)}{U(S)} \quad \frac{X(S)}{U(S)}$$

$$\frac{X(S)}{U(S)} \times \frac{Y(S)}{X(S)} = \left[\frac{1}{S^2+3S+2}\right](S+3)$$

$$\frac{X(S)}{U(S)} = \frac{1}{S^2+3S+2} \quad \text{and} \quad \frac{Y(S)}{X(S)} = S+3$$

$$Y(S) = S\,X(S) + 3X(S)$$

$U(S) = [S^2 + 3S + 2] X(S)$

$U(S) = S^2 X(S) + 3S X(S) + 2X(S)$

Taking inverse laplace Transform

$u(t) = \ddot{x} + 3\dot{x} + 2x$

choose state variables

$$\boxed{\begin{aligned} x_1 &= x \\ \dot{x}_1 &= \dot{x} = \dot{x}_1 \\ x_2 & \\ \dot{x}_2 &= \ddot{x} \end{aligned}}$$

$\dot{x} = Ax + BU$

~~$u(t) = \dot{x}_2 + 3x_1 +$~~

$u(t) = \dot{x}_2 + 3x_2 + 2x_1$

$\dot{x}_2 = -2x_1 - 3x_2 + u(t)$

$\boxed{\dot{x}_1 = x_2}$

$y(t) = x_2 + 3x_1$

$y(t) = 3x_1 + x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observable canonical form (OCF)

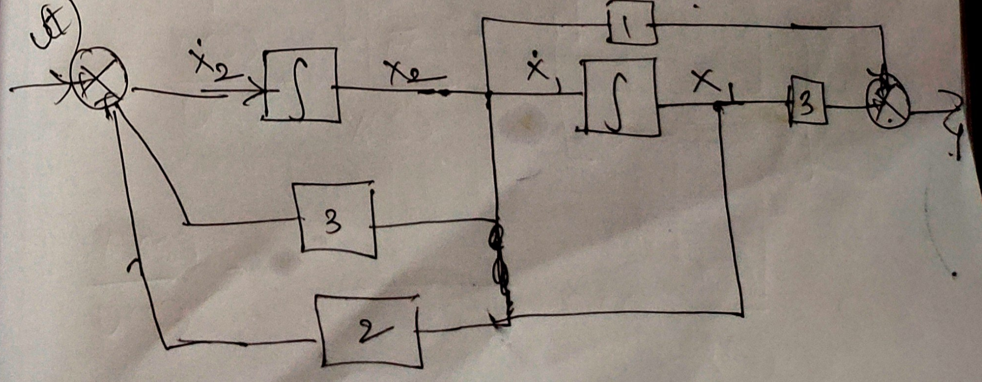
$A \rightarrow$ Replace by A^T

$B \rightarrow C^T$

$C \rightarrow B^T$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} U$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Input is force

$f(t) = u_1$

$$\boxed{u_1 = K x_1 + B x_2 + M \dot{x}_2}$$

→ for state space matrix

$\dot{x}_1 = ?$

$\dot{x}_2 = ?$

$M \dot{x}_2 = u_1 - K x_1 - B x_2$

$$\boxed{\dot{x}_2 = \frac{u_1}{M} - \frac{K}{M} x_1 - \frac{B}{M} x_2}$$

Re arrange

$$\boxed{\dot{x}_2 = -\frac{K}{M} x_1 - \frac{B}{M} x_2 + \frac{u_1}{M_1}}$$

$$\boxed{f(t) = \underset{\substack{\downarrow \\ \text{Spring}}}{Kx(t)} + \underset{\substack{\downarrow \\ \text{resistive force}}}{B\frac{dx(t)}{dt}} + \underset{\substack{\downarrow \\ \text{accelerative force}}}{M\frac{d^2x(t)}{dt}}}$$

$x(t) \rightarrow$ distance

$V(t) \rightarrow$ differentials of $x(t)$

$u(t) \rightarrow \frac{dV(t)}{dt}$

Consider state space variables

① $x(t) = x_1$

② $v(t) = \frac{dx(t)}{dt} = \dot{x}_1 = x_2$

③ $a(t) = \frac{d^2x(t)}{dt} = \frac{dV(t)}{dt} = \dot{x}_2 =$