THEORY OF MACHINES-II

(18MEC312)

Class: III year I Semester **Branch:** Mechanical Engg.

QUESTION BANK

Prepared by

Mr. N. SATHISH KUMAR

Assistant Professor

Department of Mechanical Engineering,

SITAMS, Chittoor

DOM

UNIT - I

Gyroscopic Couple and precessional motion

Introduction

When a body moves along a curved path with a uniform linear velocity, a force in the direction of centripetal acceleration (known as centripetal force - converging force) has to be applied externally over the body, so that it moves along the required path. This external force applied is known as active force.

Ex:- When a stone tied at one end of a string which whirled in a circle, the pull in the string provides the centripetal force.

→ The moon, artificial satellites which move around the earth works on this principle only.

The magnitude of the centripetal force, fc, required to cause an obj of mass m and speed v to travel in a circular path of radius r is given by the relation

$$f_c = \frac{mv^2}{r}$$

2. When a body, itself, is moving with uniform linear velocity along a circular path, it is subjected to the centrifugal force (diverging) radially outwards. This centrifugal force is called reactive force. The action of the reactive or centrifugal force is to tilt or move the body along radially outward direction.

Note :- Whenever the effect of any force or couple over a moving or rotating body is to be considered, it should be w.r.to the reactive force or couple and not w.r.to active force or couple.

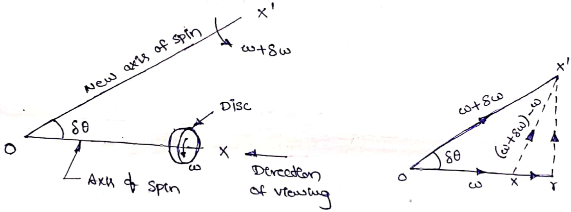
Precessional Angular Motion

The slow movement of the axis of a spinning body around another axis.

The angular acceleration is the rate of change of angular velocity w.r. to time. It is a vector quantity and may be represented by drawing a vector diagram with the help of right hand screw rule.

$$\text{Angular acceleration } \alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

vector quantity is known as it has magnitude and direction.



Consider a disc as shown in fig. revolving or spinning about the axis OX (known as axis of spin) in anticlockwise direction when seen from the front, with an angular velocity ω in a plane at right angles to the paper.

After a short interval of time δt, let the disc be spinning about the new axis of spin OX' (at an angle δθ) with an angular velocity (ω+δω). Using the right hand screw rule, initial angular velocity of the disc (ω) is represented by vector ox, and the final angular velocity of the disc (ω+δω) is represented by vector ox' as shown in fig. The vector xx' represents the change of angular velocity in time δt ie the angular acceleration of the disc. This may be resolved into two components, one parallel to ox and the other perpendicular to ox.

① Component of angular acceleration in the direction of ox,

$$\alpha_t = \frac{xr}{\delta t} = \frac{or - ox}{\delta t} = \frac{ox' \cos\delta\theta - ox}{\delta t}$$

$$= \frac{(\omega+\delta\omega)\cos\delta\theta - \omega}{\delta t}$$

$$= \frac{\omega\cos\delta\theta + \delta\omega\cos\delta\theta - \omega}{\delta t}$$

Since $\delta\theta$ is very small, $\therefore$ $\cos\delta\theta = 1$, we have

$$\alpha_t = \frac{\omega + \delta\omega - \omega}{\delta t}$$

$$= \frac{\delta\omega}{\delta t}$$

In the limit, when $\delta t \to 0$,

$$\alpha_t = \underset{\delta t \to 0}{Lt}\left(\frac{\delta\omega}{\delta t}\right) = \frac{d\omega}{dt}$$

② Component of angular acceleration in the direction perpendicular to ox,

$$\alpha_c = \frac{rx'}{\delta t} = \frac{ox' \sin\delta\theta}{\delta t}$$

$$= \frac{(\omega+\delta\omega)\sin\delta\theta}{\delta t}$$

$$= \frac{\omega\sin\delta\theta + \delta\omega\sin\delta\theta}{\delta t}$$

Since $\delta\theta$ is very small, therefore substituting $\sin\delta\theta = \delta\theta$, we have

$$\alpha_c = \frac{\omega\cdot\delta\theta + \delta\omega\cdot\delta\theta}{\delta t}$$

$$\alpha_c = \frac{\omega\cdot\delta\theta}{\delta t}$$ [Neglecting $\delta\omega\cdot\delta\theta$ being very small]

In the limit when $\delta t \to 0$,

$$\alpha_c = \underset{\delta t \to 0}{Lt}\ \frac{\omega\cdot\delta\theta}{\delta t} = \omega \times \frac{d\theta}{dt}$$

$$= \omega \cdot \omega_p \quad \left[\frac{d\theta}{dt} = \omega_p\right]$$

∴ Total angular acceleration of the disc

$$= \text{vector } xx' = \text{vector sum of } \alpha_t \text{ and } \alpha_c$$

$$= \frac{d\omega}{dt} + \omega \cdot \frac{d\theta}{dt}$$

$$= \frac{d\omega}{dt} + \omega \cdot \omega_p .$$

Where $\frac{d\theta}{dt}$ is the angular velocity of the axis of spin about a certain axis, which is perpendicular to the plane in which the axis of spin is going to rotate. This angular velocity of the axis of spin (ie $d\theta/dt$) is known as "<u>angular velocity of precession</u>" and is denoted by ω_p.

The axis, about which the axis of spin is to turn, is known as "<u>axis of precession</u>".

The angular motion of the axis of spin about the axis of precession is known as "<u>precessional angular motion</u>".

<u>Note</u>:-

1. The axis of precession is ⊥le to the plane in which the axis is going to rotate.
2. If the angular velocity of the disc remains const at all positions of the axis of spin, then $d\theta/dt$ is zero, and thus α_c is zero.
3. If the angular velocity of the disc changes the direction, but remains const in magnitude, then angular acceleration of the disc is given by

$$\alpha_c = \omega \cdot \frac{d\theta}{dt} = \omega \cdot \omega_p$$

The angular acceleration α_c is known as "<u>gyroscopic acceleration</u>".

...c couple

A device, used to provide stability or maintain a fixed direction, consisting of a wheel or disc spinning rapidly about an axis which is itself free to alter in direction.

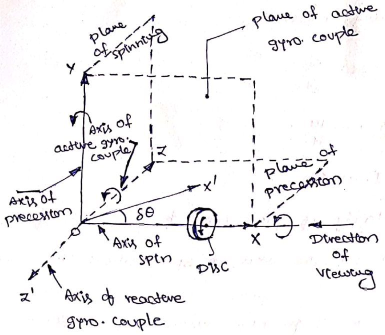
Consider a disc spinning with an angular velocity ω rad/s about the axis of spin OX, in anticlockwise direction when seen from the front as shown in fig.

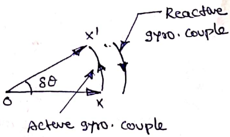
Since the plane in which the disc is rotating is parallel to the plane YOZ, therefore it is called "plane of spinning".

The plane XOZ is a horizontal plane and the axis of spin rotates in a plane parallel to the horizontal plane about an axis OY.

In other words, the axis of spin is said to be rotating or processing about an axis OY.

In other words, the axis of spin is said to be rotating or processing about an axis OY which is ⊥le to both the axes OX and OZ at an angular velocity ω_p rad/s. This horizontal plane XOZ is called "plane of precession" and OY is the "axis of precession".





Let I = Mass moment of inertia of the disc

ω = Angular velocity of the disc.

$\therefore$ Angular momentum of the disc = $I \cdot \omega$

Since the angular momentum is a vector quantity, therefore it may be represented by the vector $\vec{ox}$, as shown in fig. The axis of spin ox is also rotating anticlockwise when seen from the top about the axis OY. Let the axis OX is turned in the plane XOZ through a small angle $\delta\theta$ radians to the position OX', in time δt seconds. Assuming the angular velocity ω to be const, the angular momentum will now be represented by vector ox'.

$\therefore$ change in angular momentum

$$= \vec{ox'} - \vec{ox} = \vec{xx'} = \vec{ox} \cdot \delta\theta$$

$$= I \cdot \omega \cdot \delta\theta$$

and rate of change of angular momentum

$$= I \times \omega \times \frac{\delta\theta}{dt}$$

Since the rate of change of angular momentum will result by the application of a couple to the disc, therefore the couple applied to the disc causing precession,

$$C = \lim_{\delta t \to 0} I \cdot \omega \times \frac{\delta\theta}{\delta t}$$

$$= I \cdot \omega \times \frac{d\theta}{dt}$$

$$= I \cdot \omega \times \omega_p$$

where ω_p = Angular velocity of precession of the axis of spin or the speed of rotation of the axis of spin about the axis of precession OY.

In S.I units, the units of C is N-m when I is in kg-m².

Therefore it

couple $I.\omega.\omega_p$, in the direction of the vector xx' is the ...ve gyroscopic couple, which has to be applied over the disc when the axis of spin is made to rotate with angular velocity ω_p about the axis of precession. The vector xx' lies in the plane XOZ or the horizontal plane.

In case of a very small displacement $\delta\theta$, the vector xx' will be $\perp^{le}$ to the vertical plane XOY. Therefore the couple causing this change in the angular momentum will lie in the plane XOY.

The vector xx' represents an anticlockwise couple in the plane XOY. Therefore, the plane XOY is called the plane of active gyroscopic couple and the axis OZ $\perp^{le}$ to the plane XOY, about which the couple acts, is called the axis of active gyroscopic couple.

2. When the axis of spin itself moves with angular velocity ω_p, the disc is subjected to reactive couple whose magnitude is same (ie $I.\omega.\omega_p$) but opposite in direction to that of active couple. This reactive couple to which the disc is subjected when the axis of spin rotates about the axis of precession is known as reactive gyroscopic couple. The axis of the reactive gyroscopic couple is represented by OZ'.

3. The gyroscopic couple is usually applied through the bearings which support the shaft. The bearings will resist equal and opposite couple.

4. The gyroscopic principle is used in an instrument or toy known as gyroscope. The gyroscopes are installed in ships in order to minimize the rolling and pitching effects of waves. They are also used in aeroplanes, monorail cars, gyrocompasses etc.

Prob:-

(1) A uniform disc of dia 300 mm and of mass 5 kg is mounted on one end of an arm of length 600 mm. The other end of the arm is free to rotate in a universal bearing. If the disc rotates about the arm with a speed of 300 r.p.m. clockwise, looking from the front, with what speed will it precess about the vertical axis?

Sol:- Given data

$d = 300\ mm$

$r = 150\ mm$

$= 0.15\ m$

$m = 5\ kg$

$l = 600\ mm = 0.6\ m$

$N = 300\ r.p.m$

$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 31.42\ rad/s.$

The mass moment of inertia of the disc, about an axis through its centre of gravity and perpendicular to the plane of disc

$I = mr^2/2 = 5 \times (0.15)^2/2 = 0.056\ kg\text{-}m^2$

and couple due to mass of disc,

$C = m.g.l = 5 \times 9.81 \times 0.6 = 29.43\ N\text{-}m$

ω_p = speed of precession

$\therefore$ couple $C = I.\omega.\omega_p$

$29.43 = 0.056 \times 31.42 \times \omega_p$

$\omega_p = \frac{29.43}{1.76} = 16.7\ rad/s$

(2) A uniform disc of 150 mm dia has a mass of 5 kg. It is mounted centrally in bearings which maintain its axle in a horizontal plane. The disc spins about its axle with a const speed of 1000 r.p.m while the axle precesses uniformly about the vertical at 60 r.p.m. The directions of rotation are as shown in fig. If the distance b/w the bearings is 100 mm. find the resultant reaction at each bearing due to mass and gyroscopic effects.

Sol^n: $d = 150$ mm or $r = 75$ mm $= 0.075$ m

$m = 5$ kg

$N = 1000$ r.p.m

$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1000}{60} = 104.7$ rad/s (anticlockwise)

$N_p = 60$ r.p.m

$\omega_p = \frac{2\pi N_p}{60} = \frac{2\pi \times 60}{60} = 6.284$ rad/s (anticlockwise)

$x = 100$ mm $= 0.1$ m.

Mass moment of inertia of the disc,

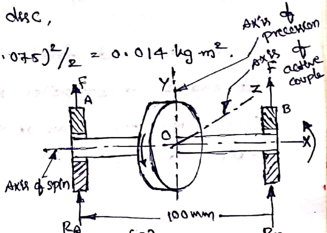
$I = mr^2/2 = 5 \times (0.075)^2/2 = 0.014$ kg m^2.

∴ Gyroscopic couple acting on the disc,

$C = I \cdot \omega \cdot \omega_p$

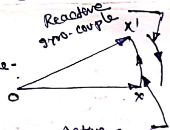
$= 0.014 \times 104.7 \times 6.284$

$= 9.2$ N-m.



(a)

The direction of the reactive gyroscopic couple is shown in fig b. Let F be the force at each bearing due to the gyroscopic couple.



(b)

$F = C/x = \frac{9.2}{0.1} = 92$ N

The force F will act in opp directions at the bearings as shown in fig a. Now let R_A and R_B be the reaction at the bearing A and B resp, due to the weight of the disc. Since the disc is mounted centrally in bearings, therefore,

$R_A = R_B = 5/2 = 2.5$ kg $= 2.5 \times 9.81 = 24.5$ N.

Resultant reaction at each bearing

Let R_{A1} and R_{B1} = Resultant reaction at the bearings A & B resp.

Scanned by CamScanner

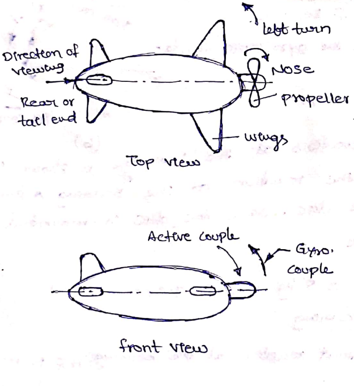
Since the reactive gyroscopic couple acts in clockwise direction when seen from the front, therefore its effect is to increase the reaction on the left hand side bearing (ie A) and to decrease the reaction on the right hand side bearing (ie B).

$\therefore$ $R_{A1} = F + R_A = 92 + 24.5 = 116.5\ N$ (upwards)

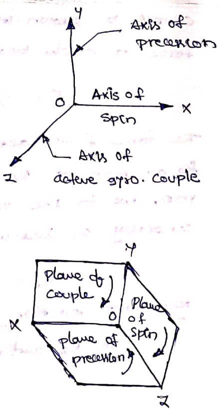
$R_{B1} = F - R_B = 92 - 24.5 = 67.5\ N$ (downwards)

Effect of the Gyroscopic Couple on an Aeroplane

The top and front view of an aeroplane are shown in fig. let engine or propeller rotates in the clockwise direction when seen from the rear or tail end and the aeroplane takes a turn to the left.



(a)



(b)

Let ω = Angular velocity of the engine in rad/s.

m = Mass of the engine and the propeller in kg.

k = Its radius of gyration in metres,

I = Mass moment of inertia of the engine and the propeller in kg-m^2.

$= m.k^2$,

V = Linear velocity of the aeroplane in m/s,

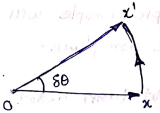
R = Radius of curvature in metres, and

ω_p = Angular velocity of precession $= \frac{V}{R}$ rad/s

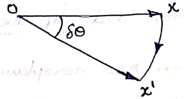
∴ Gyroscopic couple acting on the aeroplane,

$$C = I.\omega.\omega_p$$

Before taking the left turn, the angular momentum vector is represented by ox. When it takes left turn, the active gyroscopic couple will change the direction of the angular momentum vector from ox to ox' as shown in fig a. The vector xx', in the limit, represents the change of angular momentum or the active gyroscopic couple and is perpendicular to ox. Thus the plane of active gyroscopic couple xoy will be perpendicular to xx', ie vertical in this case as shown in fig. b



(a) Aeroplane taking left turn



(b) Aeroplane taking right turn.

By applying right hand screw rule to vector xx', we find that the direction of active gyroscopic couple is clockwise as shown in the front view of fig a.

In other words, for left hand turning, the active gyroscopic couple on the aeroplane in the axis OZ will be clockwise as shown in fig b. The reactive gyroscopic couple (equal in magnitude of active gyroscopic couple) will act in the opposite direction (i.e. in the anticlockwise direction) and the effect of this couple is therefore to raise the nose and dip the tail of the aeroplane.

Note :- (1) when the aeroplane takes a right turn under similar conditions as discussed above, the effect of the reactive gyroscopic couple will be to dip the nose and raise the tail of the aeroplane.

(2) when the engine or propeller rotates in anticlockwise direction when viewed from the rear or tail end and the aeroplane takes a left turn, then the effect of reactive gyroscopic couple will be to dip the nose and raise the tail of the aeroplane.

(3) when the aeroplane takes a right turn under similar conditions as mentioned in note 2 above, the effect of reactive gyroscopic couple will be to raise the nose and dip the tail of the aeroplane.

(4) when the engine or propeller rotates in clockwise direction when viewed from the front and the aeroplane takes a left turn, then the effect of reactive gyroscopic couple will be to raise the tail and dip the nose of the aeroplane.

(5) when the aeroplane takes a right turn under similar conditions as mentioned in note 4 above, the effect of reactive gyroscopic couple will be to raise the nose and dip the tail of the aeroplane.

Prob: An aeroplane makes a complete half circle of 50 mts radius, towards left, when flying at 200 km per hr. The rotary engine and the propeller of the plane has a mass of 400 kg and a radius of gyration of 0.3 m. The engine rotates at 2400 r.p.m. clockwise when viewed from the rear. Find the gyroscopic couple on the aircraft and state its effect on it.

Sol:- Given $R = 50$ m

$V = 200$ km/hr $= \frac{200 \times 10^3}{60}$ m/s $= 55.6$ m/s.

$m = 400$ kg

$k = 0.3$ m

$N = 2400$ r.p.m or $\omega = \frac{2\pi \times 2400}{60} = 251$ rad/s.

Mass moment of inertia of the engine and the propeller,

$$I = m \cdot k^2$$

$$= 400 (0.3)^2 = 36 \text{ kg-m}^2$$

Angular velocity of precession,

$$\omega_p = V/R = 55.6/50 = 1.11 \text{ rad/s}$$

Gyroscopic couple acting on the aircraft,

$$C = I \cdot \omega \cdot \omega_p$$

$$= 36 \times 251.4 \times 1.11$$

$$= 10046 \text{ N-m}$$

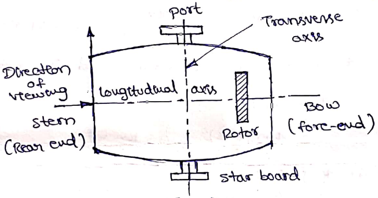
$$= 10.046 \text{ kN-m}.$$

When the aeroplane turns towards left, the effect of the gyroscopic couple is to lift the nose upwards and tail downwards.

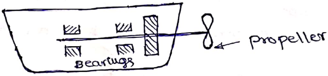
Terms Used in a Naval ship

The top and front views of a naval ship are shown in fig. The fore end of the ship is called bow and the rear end is known as stern or aft. The left hand and right hand sides of the ship, when viewed from the stern are called port and star-board resp. we shall now discuss the effect of gyroscopic couple on the naval ship in the following three cases.

(1) steering (2) pitching (3) Rolling.



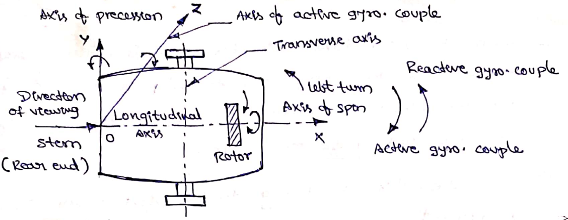
Top view



front view

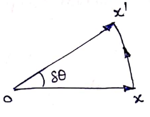
Effect of Gyroscopic Couple on a Naval ship during steering

Steering is the turning of a complete ship in a curve towards left or right, while it moves forward. Consider the ship taking a left turn, and rotor rotates in the clockwise direction when viewed from the stern, as shown in fig. The effect of gyroscopic couple on a naval ship during steering taking left or right turn may be obtained in the similar way as for an aeroplane.

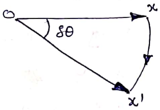


Naval ship taking a left turn

When the rotor of the ship rotates in the clockwise direction when viewed from the stern, it will have its angular momentum vector in the direction ox as shown in fig a. As the ship steers to the left, the active gyroscopic couple will change the angular momentum vector from ox to ox'. The vector xx' now represents the active gyroscopic couple and is $\perp^{le}$ to ox. Thus the plane of active gyroscopic couple is $\perp^{le}$ to xx' and its direction in the axis oz for left hand turn is clockwise as shown in fig. The reactive gyroscopic couple of the same magnitude will act in the opposite direction (ie in anticlockwise direction). The effect of this reactive gyroscopic couple is to raise the bow and lower the stern.



steering to the left



steering to the right.

Notes (1) When the ship steers to the right under similar conditions as discussed above, the effect of the reactive gyroscopic couple as shown in fig b. will be to raise the stern and lower the bow.

(2) When the rotor rotates in the anticlockwise direction, when viewed from the stern and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to lower the bow and raise the stern.

(3) When the ship is steering to the right under similar conditions as discussed in note 2, then the effect of reactive gyroscopic couple will be to raise the bow and lower the stern.

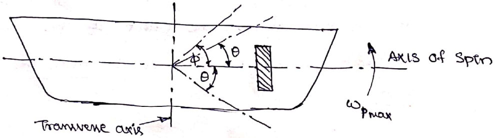
(4) When the rotor rotates in the clockwise direction when viewed from the bow or fore end and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to raise the stern and lower the bow.

(5) When the ship is steering to the right under similar conditions as discussed in note 4, then the effect of reactive gyroscopic couple will be to raise the bow and lower the stern.

(6) The effect of the reactive gyroscopic couple on a boat propelled by a turbine taking left or right turn is similar as discussed above.

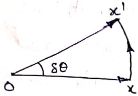
Effect of Gyroscopic Couple on a Naval ship during pitching

Pitching is the movement of a complete ship up and down in a vertical plane about transverse axis as shown in fig. (a). In this case, the transverse axis is the axis of precession. The pitching of the ship is assumed to

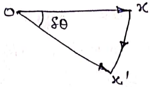


(a) Pitching of a naval ship

take place with Simple Harmonic motion i.e. the motion of the axis of spin about transverse axis is simple harmonic.



(b) Pitching upward



(c) Pitching downward.

∴ Angular displacement of the axis of spin from mean position after time t seconds.

$$\theta = \phi \sin \omega_1 \cdot t$$

where ϕ = Amplitude of swing i.e. max angle turned from the mean position in radians, and

ω_1 = Angular velocity of S.H.M

$$= \frac{2\pi}{\text{Time period of S.H.M in sec}} = \frac{2\pi}{t_p} \text{ rad/s}$$

Angular velocity of precession,

$$\omega_p = \frac{d\theta}{dt} = \frac{d}{dt}(\phi \sin \omega_1 \cdot t)$$

$$= \phi \omega_1 \cos \omega_1 t$$

The angular velocity of precession will be max, if $\cos \omega_1 \cdot t = 1$

∴ Max angular velocity of precession,

$$\omega_{P_{max}} = \phi \cdot \omega_1 = \phi \cdot 2\pi / t_p$$

let I = Moment of inertia of the rotor in $kg\text{-}m^2$, and

ω = Angular velocity of the rotor in rad/s.

∴ Max gyroscopic couple,

$$C_{max} = I \cdot \omega \cdot \omega_{P_{max}}$$

When the pitching is upward, the effect of the reactive gyroscopic couple, as shown in fig b. will try to move the ship toward star board. On the other hand, if the pitching is downward, the effect of the reactive gyroscopic couple, as shown in fig c. is to turn the ship towards port side.

Note :- (1) The effect of the gyroscopic couple is always given on specific position of the axis of spin i.e whether it is pitching downwards or upwards.

(2) The pitching of a ship produces forces on the bearings which act horizontally and perpendicular to the motion of the ship.

(3) The angular acceleration during pitching,

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right)$$

$$= \frac{d}{dt}\left(\phi\, \omega_1 \cos \omega_1 t\right)$$

$$= -\phi \cdot \omega_1^2 \cdot \sin \omega_1 \cdot t$$

The angular acceleration is max, if $\sin \omega_1 t = 1$

∴ Max angular acceleration during pitching,

$$\alpha_{max} = (\omega_1)^2$$

Effect of Gyroscopic couple on a Naval ship during Rolling

The effect of gyroscopic couple to occur, the axis of precession should always be perpendicular to the axis of spin. If, however, the axis of precession becomes parallel to the axis of spin, there will be no effect of the gyroscopic couple acting on the body of the ship.

In case of rolling of a ship, the axis of precession is always parallel to the axis of spin for all positions. Hence, there is no effect of the gyroscopic couple acting on the body of a ship.

Prob: The turbine rotor of a ship has a mass of 8 tonnes and a radius of gyration 0.6 m. It rotates at 1800 r.p.m. clockwise, when looking from the stern. Determine the gyroscopic couple, if the ship travels at 100 km/h and steer to the left in a curve of 75 m radius.

Sol:- Given $m = 8t = 8000\ kg$; $k = 0.6m$, $N = 1800$ r.p.m or $\omega = \frac{2\pi N}{60}$

$V = 100\ km/h = 27.8\ m/s$ $= \frac{2\pi \times 1800}{60}$

$R = 75\ m.$ $= 188.5\ rad/s.$

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 8000(0.6)^2 = 2880\ kg\text{-}m^2$$

and angular velocity of precession,

$$\omega_p = V/R = 27.8/75 = 0.37\ rad/s.$$

Gyroscopic couple,

$$C = I.\omega.\omega_p = 2880 \times 188.5 \times 0.37 = 200\ 866\ N\text{-}m$$

$$= 200.866\ kN\text{-}m.$$

When the rotor rotates in clockwise direction when looking from the stern and the ship steers to the left, the effect of the reactive gyroscopic couple is to raise the bow and lower the stern.

Prob:- The heavy turbine rotor of a sea vessel rotates at 1500 r.p.m. clockwise looking from the stern, its mass being 750 kg. The vessel pitches with an angular velocity of 1 rad/s. Determine the gyroscopic couple transmitted to the hull when bow is rising, if the radius of gyration for the rotor is 250 mm. Also show in what direction the couple acts on the hull?

Sol:- Given $N = 1500$ r.p.m or $\omega = \frac{2\pi \times 1500}{60} = 157.1$ rad/s

$m = 750$ kg.

$\omega_p = 1$ rad/s,

$k = 250$ mm $= 0.25$ m

Mass moment of inertia of the rotor,

$I = m \cdot k^2 = 750 (0.25)^2 = 46.875$ kg-m^2

∴ Gyroscopic couple transmitted to the hull (i.e body of the sea vessel)

$C = I \cdot \omega \cdot \omega_p = 46.875 \times 157.1 \times 1$

$= 7364$ N-m

$= 7.364$ kN-m.

When the bow is rising i.e when the pitching is upward, the reactive gyroscopic couple acts in the clockwise direction which moves the sea vessel towards star-board.

Prob:- The turbine rotor of a ship has a mass of 3500 kg. It has a radius of gyration of 0.45 m and a speed of 3000 r.p.m. clockwise when looking from stern. Determine the gyroscopic couple and its effect upon the ship:

(1) When the ship is steering to the left on a curve of 100 m radius at a speed of 36 km/h.

(2) When the ship is pitching in a simple harmonic motion, the bow falling with its max velocity. The period of pitching is 40 sec and the total angular displacement b/w the two extreme positions of pitching is 12 degrees.

Sol:- Given: $m = 3500$ kg, $k = 0.45$ m, $N = 3000$ r.p.m or $\omega = \frac{2\pi \times 3000}{60}$

$= 314.2$ rad/s

① When the ship is steering to the left

Given $R = 100 m$, $V = 36 km/h = 10 m/s$

Mass moment of inertia of the rotor,

$$I = m.k^2 = 3500 (0.45)^2 = 708.75 \ kg\text{-}m^2$$

and angular velocity of precession

$$\omega_p = V/R = 10/100 = 0.1 \ rad/s.$$

∴ Gyroscopic couple,

$$C = I.\omega.\omega_p = 708.75 \times 314.2 \times 0.1 = 22270 \ N\text{-}m.$$

$$= 22.27 \ kN\text{-}m.$$

When the rotor rotates clockwise when looking from the stern and the ship takes a left turn, the effect of the reactive gyroscopic couple is to raise the bow and lower the stern.

② When the ship is pitching with the bow falling

Given $t_p = 40 s$.

Since the total angular displacement b/w the two extreme positions of pitching is 12° ie $2\phi = 12°$.

∴ Amplitude of swing, $\phi = 12/2 = 6° = \frac{6 \times \pi}{180} = 0.105 \ rad$

Angular velocity of the S.H.M

$$\omega_1 = 2\pi / t_p = 2\pi / 40 = 0.157 \ rad/s.$$

max angular velocity of precession,

$$\omega_p = \phi.\omega_1 = 0.105 \times 0.157 = 0.0165 \ rad/s.$$

∴ Gyroscopic couple, $C = I.\omega.\omega_p$

$$= 708.75 \times 314.2 \times 0.0165 = 3675 \ N\text{-}m$$

$$= 3.675 \ kN\text{-}m$$

When the bow is falling (i.e. when the pitching is downward) the effect of the reactive gyroscopic couple is to move the ship towards port side.

Prob: The mass of the turbine rotor of a ship is 20 tonnes and has a radius of gyration of 0.60 m. Its speed is 2000 r.p.m. The ship pitches 6° above and 6° below the horizontal position. A complete oscillation takes 30 seconds and the motion is simple harmonic. Determine the following:
(1) Max gyroscopic couple (2) Max angular acceleration of the ship during pitching, and (3) The direction in which the bow will tend to turn when rising, if the rotation of the rotor is clockwise when looking from the left.

Sol:- Given $m = 20t = 20 \times 10^3$ kg, $k = 0.6$ m, $N = 2000$ r.p.m or $\omega = \frac{2\pi \times 2000}{60} = 209.5$ rad/s

$\phi = 6° = \frac{6 \times \pi}{180} = 0.105$ rad, $t_p = 30s$

1. Max gyroscopic couple

max moment of inertia of the rotor,

$$I = mk^2 = 20\,000(0.6)^2 = 7200 \text{ kg-m}^2$$

Angular velocity of the simple harmonic motion

$$\omega_1 = 2\pi/t_p = 2\pi/30 = 0.21 \text{ rad/s}$$

∴ Max angular velocity of precession,

$$\omega_{P\,max} = \phi . \omega_1 = 0.105 \times 0.21 = 0.022 \text{ rad/s}$$

max gyroscopic couple,

$$C_{max} = I . \omega . \omega_{P\,max} = 7200 \times 209.5 \times 0.022$$
$$= 33185 \text{ N-m}$$
$$= 33.185 \text{ kN-m}$$

(2) Max angular acceleration during pitching

$$= \phi(\omega_1)^2 = 0.105(0.21)^2 = 0.0046 \text{ rad/s}^2.$$

(3) When the rotation of the rotor is clockwise when looking from the left (ie rear end or stern) and when the bow is rising (ie pitching is upward) then the reactive gyroscopic couple acts in the clockwise direction which tends to turn the bow towards right (ie towards star-board).

Stability of a four wheel Drive Moving in a curved path

Consider the four wheels A, B, C and D of an automobile locomotive taking a turn towards left as shown in fig. The wheels A and C are inner wheels, whereas B and D are outer wheels. The centre of gravity (C.G) of the vehicle lies vertically above the road surface.

Let m = Mass of the vehicle in kg.

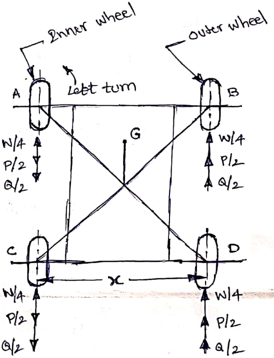
W = Weight of the vehicle in newtons = $m \cdot g$.

r_w = Radius of the wheels in mts.

R = Radius of curvature in mts ($R > r_w$)

h = Distance of centre of gravity, vertically above the road surface in mts.

x = Width of track in mts.



I_W = Mass moment of inertia of one of the wheels in kg-m²,

ω_W = Angular velocity of the wheels or velocity of spin in rad/s

I_E = Mass moment of inertia of the rotating parts of the engine in kg-m²,

ω_E = Angular velocity of the rotating parts of the engine in rad/s.

G = Gear ratio = ω_E / ω_W

v = Linear velocity of the vehicle in m/s = $r_w \cdot \omega_W$

A little consideration will show, that the weight of the vehicle (W) will be equally distributed over the four wheels which will act downwards. The reaction b/w each wheel and the road surface of the same magnitude will act upwards. Therefore, Road reaction over each wheel $= \frac{W}{4} = \frac{m \cdot g}{4}$ newtons.

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

1. Effect of the gyroscopic couple

Since the vehicle takes a turn towards left due to the precession and other rotating parts, therefore a gyroscopic couple will act.

We know that velocity of precession,

$$\omega_P = V/R$$

$\therefore$ Gyroscopic couple due to 4 wheels,

$$C_W = 4 \cdot I_W \cdot \omega_W \cdot \omega_P$$

and gyroscopic couple due to the rotating parts of the engine

$$C_E = I_E \cdot \omega_E \cdot \omega_P$$

$$= I_E \cdot G \omega_W \cdot \omega_P \qquad (\because G = \omega_E / \omega_W)$$

$\therefore$ Net gyroscopic couple,

$$C = C_W \pm C_E$$

$$= 4 I_W \cdot \omega_W \cdot \omega_P \pm I_E \cdot G \omega_W \cdot \omega_P$$

$$= \omega_W \cdot \omega_P (4 I_W \pm G I_E)$$

The positive sign is used when the wheels and rotating parts of the engine rotate in the same direction. If the rotating parts of the engine revolves in opp direction, then negative sign is used.

Due to the gyroscopic couple, vertical reaction on the road surface will be produced. The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be P newtons. Then

$$P \times x = C \quad \text{or} \quad P = C/x$$

∴ Vertical reaction at each of the outer or inner wheels,

$$P/2 = C/2x$$

Note :- When rotating parts of the engine rotate in opposite directions, then -ve sign is used, i.e. net gyroscopic couple,

$$C = C_W - C_E$$

When $C_E > C_W$, then C will be -ve. Thus the reaction will be vertically downwards on the outer wheels and vertically upwards on the inner wheels.

(2) Effect of the centrifugal Couple

Since the vehicle moves along a curved path, therefore centrifugal force will act outwardly at the centre of gravity of the vehicle. The effect of this centrifugal force is also to overturn the vehicle.

We know that centrifugal force,

$$F_C = \frac{m \times v^2}{R}$$

∴ The couple tending to overturn the vehicle or overturning Couple,

$$C_O = F_C \times h = \frac{m \cdot v^2}{R} \times h$$

This overturning couple is balanced by vertical reactions, which are vertically upwards on the outer wheels and vertically down-wards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be Q. Then

$$Q \times x = C_O \quad \text{or} \quad Q = \frac{C_O}{x} = \frac{m \cdot v^2 h}{R \cdot x}$$

∴ vertical reaction at each of the outer or inner wheels,

$$\frac{Q}{2} = \frac{m v^2 h}{2R \cdot x}$$

∴ Total vertical reaction at each of the outer wheel,

$$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

Total vertical reaction at each of the inner wheel,

$$P_i = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

A little consideration will show that when the vehicle is running at high speeds, P_i may be zero or even negative. This will cause the inner wheels to leave the ground thus tending to overturn the automobile. In order to have the contact b/w the inner wheels and the ground, the sum of $P/2$ and $Q/2$ must be less than $W/4$.

Probl: A rear engine automobile is travelling along a track of 100 mts mean radius. Each of the four road wheels has a moment of inertia of 2.5 kg-m² and an effective dia of 0.6m. The rotating parts of the engine have a moment of inertia of 1.2 kg-m². The engine axis is parallel to the rear axle and the crankshaft rotates in the same sense as the road wheels. The ratio of engine speed to back axle speed is 3:1. The automobile has a mass of 1600 kg and has its centre of gravity 0.5m above road level. The width of the track of the vehicle is 1.5 m.

Determine the limiting speed of the vehicle around the curve for all four wheels to maintain contact with the road surface. Assume that the road surface is not cambered and centre of gravity of the automobile lies centrally w.r.t the four wheels.

Sol:- Given: $R = 100m$, $I_W = 2.5\ kg\text{-}m^2$, $d_W = 0.6m$ or $r_W = 0.3m$,

$I_E = 1.2\ kg\text{-}m^2$, $G = \omega_E/\omega_W = 3$, $m = 1600\ kg$, $h = 0.5m$, $x = 1.5m$

The wt of the vehicle (m·g) will be equally distributed over the four wheels which will act downwards. The reaction b/w the wheel and the road surface of the same magnitude will act upwards.

∴ Road reaction over each wheel

$$= W/4 = m \cdot g/4 = \frac{1600 \times 9.81}{4} = 3924\ N$$

let v = Limiting speed of the vehicle in m/s.

Angular velocity of the wheels,

$$\omega_W = \frac{v}{r_W} = \frac{v}{0.3} = 3.33v\ rad/s.$$

Angular velocity of precession,

$$\omega_P = \frac{v}{R} = \frac{v}{100} = 0.01 v \text{ rad/s}.$$

∴ Gyroscopic couple due to 4 wheels,

$$C_W = 4 I_W \cdot \omega_W \cdot \omega_P$$

$$= 4 \times 2.5 \times \frac{v}{0.3} \times \frac{v}{100}$$

$$= 0.33 v^2 \text{ N-m}$$

and gyroscopic couple due to rotating parts of the engine,

$$C_E = I_E \cdot \omega_E \cdot \omega_P$$

$$= I_E \cdot G\, \omega_W \cdot \omega_P$$

$$= 1.2 \times 3 \times 3.33 v \times 0.01 v$$

$$= 0.12 v^2 \text{ N-m}$$

∴ Total gyroscopic couple,

$$C = C_W + C_E = 0.33 v^2 + 0.12 v^2$$

$$= 0.45 v^2 \text{ N-m}$$

Due to this gyroscopic couple, the vertical reaction on the road surface will be produced. The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheel be P/2 newtons,

$$P/2 = \frac{C}{2x} = \frac{0.45 v^2}{2 \times 1.5} = 0.15 v^2 \text{ N}$$

we know that centrifugal force,

$$F_C = \frac{m \cdot v^2}{R} = \frac{1600 \times v^2}{100} = 16 v^2 \text{ N}$$

∴ overturning couple acting in the outward direction

$$C_o = F_C \times h = 16 v^2 \times 0.5 = 8 v^2 \text{ N-m}$$

∴ The overturning couple is balanced by vertical reactions which are vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheels be $Q/2$ newtons.

$$Q/2 = C_o/2x = \frac{8v^2}{2\times1.5} = 2.67v^2 \text{ N}$$

We know that total vertical reaction at each of the outer wheels,

$$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2} \longrightarrow (1)$$

and total vertical reaction at each of the inner wheels,

$$P_i = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

$$= \frac{W}{4} - \left(\frac{P}{2} + \frac{Q}{2}\right) \longrightarrow (2)$$

From eqn (1), we see that there will always be contact b/w the outer wheels and the road surface, because $\frac{W}{4}$, $\frac{P}{2}$ & $Q/2$ are vertically upwards. In order to have contact b/w the inner wheels and road surface, the reactions should also be vertically upwards, which is only possible if

$$\frac{P}{2} + \frac{Q}{2} \leq \frac{W}{4}$$

$$0.15v^2 + 2.67v^2 \leq 3924$$

$$2.82v^2 \leq 3924$$

$$v^2 \leq 3924/2.82 = 1391.5$$

$$v \leq 37.3 \text{ m/s}$$

$$= \frac{37.3 \times 3600}{1000} = 134.28 \text{ km/h}$$

Prob:- A four wheeled motor car of mass 2000 kg has a wheel base 2.5 m, track width 1.5 m and height of C.G 500mm above the Ground level and lies at 1 mt from the front axle. Each wheel has an effective dia of 0.8 m and a moment of inertia of 0.8 $kg \cdot m^2$. The drive shaft, engine flywheel and transmission are rotating at 4 times the speed of road wheel, in a clockwise direction when viewed from the front, and is equivalent to a mass of 75 kg having a radius of gyration of 100 mm. If the car is taking a right turn of 60 m radius at 60 km/h. Find the load on each wheel.

Sol:- Given data, $m = 2000$ kg, $b = 2.5$ m, $x = 1.5$ m, $h = 500$ mm $= 0.5$ m,

$L = 1$m, $d_W = 0.8$ m or $r_W = 0.4$ m, $I_W = 0.8$ $kg \cdot m^2$, $G = \frac{\omega_E}{\omega_W} = 4$,

$m_E = 75$ kg, $k_E = 100$ mm $= 0.1$ m, $R = 60$ m, $V = 60$ km/h $= 16.67$ m/s

$= \frac{60 \times 5}{18} =$

Since the C.G of the car lies at 1m from the front axle and the w.t of the car ($W = m \cdot g$) lies at the centre of gravity, therefore w.t on the front wheels and rear wheels will be different.

Let $W_1 =$ w.t on the front wheels,

$W_2 =$ w.t on the rear wheels.

Taking moment about the front wheels,

$W_2 \times 2.5 = W \times 1 = m \cdot g \times 1 = 2000 \times 9.81 \times 1 = 19620$

$W_2 = 19620 / 2.5 = 7848$ N

w.t of the car or on the four wheels,

$W = W_1 + W_2$

$W_1 = W - W_2 = 19620 - 7848 = 11772$ N

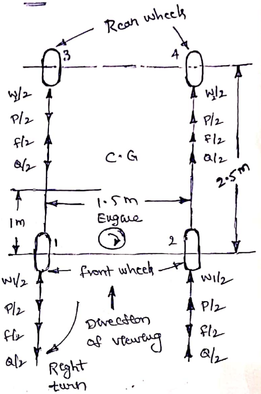
∴ w.t on each of the front wheels

$= W_1/2 = 11772/2 = 5886$ N

∴ W.t on each of the rear wheels

$= W_2/2 = 7848/2 = 3924$ N

Since the w.t of the car over the four wheels will act downwards, therefore the reaction b/w each wheel and the road surface of the same magnitude will act upwards as shown in fig.



Let us now consider the effect of gyroscopic couple due to four wheels and rotating parts of the engine.

We know angular velocity of wheels,

$$\omega_w = \frac{v}{r_w} = \frac{16.67}{0.4} = 41.675 \text{ rad/s}$$

Angular velocity of precession,

$$\omega_p = v/R = \frac{16.67}{60} = 0.278 \text{ rad/s}$$

∴ Gyroscopic couple due to four wheels,

$$C_w = 4 I_w \cdot \omega_w \cdot \omega_p$$

$$= 4 \times 0.8 \times 41.675 \times 0.278 = 37.1 \text{ N-m}$$

This gyroscopic couple tends to lift the inner wheels and to press the outer wheels. In other words, the reaction will be vertically downward on the inner wheels (i.e wheels 1 & 3) and vertically upward on the outer wheels (i.e wheels 2 & 4) as shown in fig.

Let P/2 newtons be the magnitude of this reaction at each of the inner or outer wheel.

$$\therefore P/2 = \frac{C_w}{2x} = \frac{37.1}{2 \times 1.5} = 12.37 \text{ N}$$

mass moment of inertia of rotating parts of the engine;

$$I_E = m_E (k_E)^2 = 75 (0.1)^2 = 0.75 \text{ kg-m}^2$$

∴ Gyroscopic couple due to rotating parts of the engine,

$$C_E = I_E \cdot \omega_E \cdot \omega_P$$

$$= I_E \times G \cdot \omega_W \cdot \omega_P$$

$$= 0.75 \times 4 \times 41.675 \times 0.278$$

$$= \underline{34.7 \text{ N-m}}$$

This gyroscopic couple tends to lift the front wheels and to press the rear wheels. In other words, the reaction will be vertically downwards on the front wheels and vertically upwards on the rear wheels as shown in fig. Let $F/2$ newtons be the magnitude of this reaction on each of the front and rear wheels.

$$\therefore \quad F/2 = C_E/2b = 34.7/2\times 2.5 = 6.94 \text{ N}$$

Now let us consider the effect of centrifugal couple acting on the car. We know that centrifugal force,

$$F_c = \frac{m \cdot V^2}{R} = \frac{2000 (16.67)^2}{60} = 9263 \text{ N}$$

∴ Centrifugal couple tending to overturn the car or overturning couple.

$$C_o = F_c \times h = 9263 \times 0.5 = \underline{4631.5 \text{ N-m}}$$

This overturning couple tends to reduce the pressure on the inner wheels and to increase on the outer wheels. In other words, the reactions are vertically downward on the inner wheels and vertically upwards on the outer wheels. Let $Q/2$ be the magnitude of this reaction on each of the inner & outer wheels.

$$\therefore \quad Q/2 = C_o/2x = \frac{4631.5}{2\times 1.5} = \underline{1543.83 \text{ N}}$$

∴ Load on the front wheel (1)

$$= \frac{W_1}{2} - \frac{P}{2} - \frac{f}{2} - \frac{Q}{2} = 5886 - 12.37 - 6.94 - 1543.83$$

$$= 4322.86 \text{ N}$$

Load on the front wheel (2)

$$= \frac{W_1}{2} + \frac{P}{2} - \frac{f}{2} + \frac{Q}{2} = 5886 + 12.37 - 6.94 + 1543.83$$

$$= 7435.26 \text{ N}$$

Load on the rear wheel (3)

$$= \frac{W_2}{2} - \frac{P}{2} + \frac{f}{2} - \frac{Q}{2} = 3924 - 12.37 + 6.94 - 1543.83$$

$$= 2374.74 \text{ N}$$

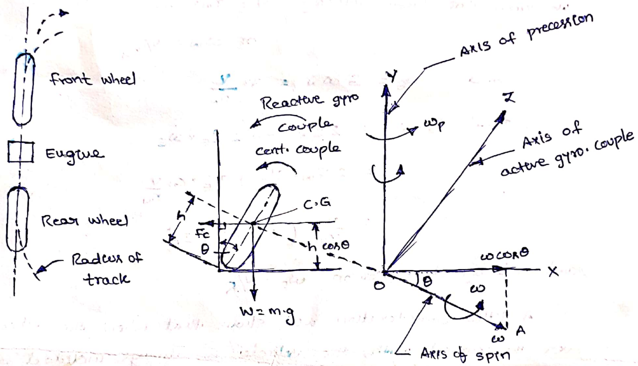
Load on the rear wheel (4)

$$= \frac{W_2}{2} + \frac{P}{2} + \frac{f}{2} + \frac{Q}{2} = 3924 + 12.37 + 6.94 + 1543.83$$

$$= 5487.14 \text{ N}$$

Stability of a two wheel vehicle taking a turn.

Consider a two wheel vehicle (scooter or motor cycle) taking a right turn as shown in fig.



Let

m = Mass of the vehicle and its rider in kg,

W = wt of the vehicle and its rider in Newtons = m.g

h = height of the C.G of the vehicle and rider.

r_W = Radius of the wheels,

R = Radius of track or curvature,

I_W = mass moment of inertia of each wheel,

I_E = Mass moment of inertia of the rotating parts of the engine.

ω_W = Angular velocity of the wheels,

ω_E = Angular velocity of the engine,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle = $r_W \times \omega_W$,

θ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium.

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle,

1. Effect of gyroscopic couple

We know that linear velocity $v = r_W \times \omega_W$

or $\omega_W = v / r_W$

and $\omega_E = G \omega_W = G \times \frac{v}{r_W}$

$\therefore$ Total $(I\omega) = 2 I_W \times \omega_W \pm I_E \times \omega_E$

$= 2 I_W \times \frac{v}{r_W} \pm I_E \times G \times \frac{v}{r_W}$

$= \frac{v}{r_W} (2 I_W \pm G I_E)$

and velocity of precession, $\omega_P = v / R$

A little consideration will show that when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in fig b. This angle is known as angle of heel.

In other words, the axis of spin is inclined to the horizontal at an angle θ, as shown in fig.(c) Thus the angular momentum vector $I\omega$ due to spin is represented by OA inclined to OX at an angle θ. But the precession axis is vertical. Therefore the spin vector is resolved along OX.

$\therefore$ Gyroscopic couple,

$C_1 = I\omega \cos\theta \times \omega_P$

$= \frac{v}{r_W} (2 I_W \pm G I_E) \cos\theta \times \frac{v}{R}$

$= \frac{v^2}{R \cdot r_W} (2 I_W \pm G \cdot I_E) \cos\theta$

Note· (1) When the engine is rotating in the same direction as that of wheels, then the +ve sign is used in the above expression and if the engine rotates in opp direction, then negative sign is used.

(2) The gyroscopic couple will act over the vehicle outwards i.e in the anticlock wise direction when seen from the front of the vehicle. The tendency of this couple is to overturn the vehicle in outward direction.

(2) Effect of centrifugal couple

We know that centrifugal force,

$$F_c = \frac{mv^2}{R}$$

This force acts horizontally through the C.G along the outward direction.

∴ centrifugal couple

$$C_2 = F_c \times h\cos\theta = \frac{mv^2}{R} h\cos\theta.$$

Since the centrifugal couple has a tendency to overturn the vehicle, therefore

Total overturning couple,

$$C_o = \text{Gyroscopic couple} + \text{Centrifugal couple}$$

$$= \frac{v^2}{R \cdot r_w}\left(2 I_w + G \cdot I_E\right)\cos\theta + \frac{m \cdot v^2}{R} \times h\cos\theta$$

$$= \frac{v^2}{R}\left[\frac{2 I_w + G I_E}{r_w} + m \cdot h\right]\cos\theta$$

We know that balancing couple = $m \cdot g \cdot h \sin\theta$

The balancing couple acts in clockwise direction when seen from the front of the vehicle, Therefore for stability, the overturning couple must be equal to the balancing couple, ie

$$\frac{v^2}{R}\left[\frac{2 I_w + G \cdot I_E}{r_w} + m \cdot h\right]\cos\theta = m \cdot g \cdot h\sin\theta$$

From this expression, the value of the angle of heel (θ) may be determined, so that the vehicle does not skid.

Prob Find the angle of inclination w.r.t the vertical of a two wheeler negotiating a turn. Given: Combined mass of the vehicle with its rider 250 kg; moment of inertia of the engine flywheel 0.3 kg-m², moment of inertia of each road wheel 1 kg-m²; Speed of engine flywheel 5 times that of road wheels and in the same direction; height of C.G of rider with vehicle 0.6 m, two wheeler speed 90 km/h, wheel radius 300 mm, radius of turn 50 m.

Sol:- Given; $m = 250$ kg, $I_E = 0.3$ kg-m², $I_W = 1$ kg-m², $\omega_E = 5\,\omega_W$ or $G = \frac{\omega_E}{\omega_W} = 5$;

$h = 0.6$ m, $v = 90$ km/h $= \frac{90 \times 5}{18} = 25$ m/s, $r_W = 300$ mm $= 0.3$ m, $R = 50$ m

let θ = Angle of inclination w.r.t the vertical of a two wheeler.

We know that gyroscopic couple,

$$C_1 = \frac{v^2}{R \cdot r_W}\left(2 I_W + G \cdot I_E\right)\cos\theta$$

$$= \frac{(25)^2}{50 \times 0.3}\left(2 \times 1 + 5 \times 0.3\right)\cos\theta$$

$$= 146 \cos\theta \text{ N-m.}$$

and centrifugal couple, $C_2 = \frac{m \cdot v^2}{R} \times h\cos\theta = \frac{250(25)^2}{50} \times 0.6\cos\theta$

$$= 1875 \cos\theta \text{ N-m.}$$

∴ Total over turning couple,

$$C = C_1 + C_2 = 146\cos\theta + 1875\cos\theta = 2021\cos\theta \text{ N-m}$$

We know that balancing couple

$$= m \cdot g \cdot h \sin\theta = 250 \times 9.81 \times 0.6 \sin\theta = 1471.5 \sin\theta \text{ N-m}$$

Since the overturning couple must be equal to the balancing couple for equilibrium condition,

$$\therefore \quad 2021\cos\theta = 1471.5\sin\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2021}{1471.5} = 1.3734 \text{ or } \theta = 53.94^\circ$$

Prob:- A four wheeled trolley car of mass 2500 kg runs on rails, which are 1.5 m apart and travels around a curve of 30 m radius at 24 km/hr. The rails are at the same level. Each wheel of the trolley is 0.75 m in diameter and each of the two axles is driven by a motor running in a direction opp to that of the wheels at a speed of 5 times the speed of rotation of the wheels. The moment of inertia of each axle with gear and wheels is 18 kg-m². Each motor with shaft and gear pinion has a moment of inertia of 12 kg-m². The C.G of the car is 0.9 m above the rail level. Determine the vertical force exerted by each wheel on the rails taking into consideration the centrifugal and gyroscopic effects. State the centrifugal and gyroscopic effects on the trolley.

Sol:- Given $m = 2500$ kg, $x = 1.5$ m, $R = 30$ m, $v = 24$ km/hr $= 24 \times \frac{5}{18} = 6.67$ m/s

$d_w = 0.75$ m or $r_w = 0.375$ m, $G = \frac{\omega_E}{\omega_W} = 5$;

$I_W = 18$ kg-m², $I_E = 12$ kg-m², $h = 0.9$ m.

The wt. of the trolley ($W = m \cdot g$) will be equally distributed over the four wheels, which will act downwards. The reaction b/w the wheels and the road surface of the same magnitude will act upwards.

∴ Road reaction over each wheel $= \frac{W}{4} = \frac{m \cdot g}{4}$

$= \frac{2500 \times 9.81}{4}$

$= 6131.25$ N

Angular velocity of the wheels,

$$\omega_W = v/r_W = \frac{6.67}{0.375} = 17.8 \text{ rad/s}$$

and Angular velocity of precession,

$$\omega_P = V/R = \frac{6.67}{30} = 0.22 \text{ rad/s}$$

∴ Gyroscopic couple due to one pair of wheels and axle,

$$C_W = 2 I_W \cdot \omega_W \cdot \omega_P = 2 \times 18 \times 17.8 \times 0.22$$

$$= 141 \text{ N-m}$$

and gyroscopic couple due to the rotating parts of the motor & gears,

$$C_E = 2 I_E \cdot \omega_E \cdot \omega_P$$

$$= 2 I_E \cdot \omega_W G \cdot \omega_P$$

$$= 2 \times 12 \times 17.8 \times 5 \times 0.22 = 470 \text{ N-m}$$

$\therefore$ net gyroscopic couple, $C = C_W - C_E = 141 - 470 = -329$ N-m.

$$P/2 = \frac{C}{2x} = \frac{329}{2 \times 1.5} = 109.7 \text{ N}$$

$$F_C = \frac{mv^2}{R} = \frac{2500 (6.67)^2}{30} = 3707 \text{ N}$$

$$C_O = F_C \times h = 3707 \times 0.9 = 3336.3 \text{ N-m}$$

$$\frac{Q}{2} = \frac{C_O}{2x} = \frac{3336.3}{2 \times 1.5} = 1112.1 \text{ N}$$

$$P_O = \frac{W}{4} - \frac{P}{2} + \frac{Q}{2} = 6131.25 - 109.7 + 1112.1 = \underline{7142.65 \text{ N}}$$

$$P_I = \frac{W}{4} + \frac{P}{2} - \frac{Q}{2} = 6131.25 + 109.7 - 1112.1 = \underline{5128.85 \text{ N}}$$

Table 3.1

Materials	Coefficient of friction μ
Steel on steel	0.54
Cast iron on steel	0.14
Wood on wood	0.27
Cast iron on cast iron	0.41
Leather on wood	0.40
Glass on glass	0.40
Metal on wood	0.2 to 0.60
Bronze on cast iron	0.23

3.4 LIMITING ANGLE OF FRICTION

In Fig. 3.2, a body B of weight W is resting on a horizontal plane S. A horizontal force F is applied to the body, no relative motion takes place until the applied force F is equal to the force of friction F'. As soon as $F > F'$, the body starts sliding on the plane S. The magnitude of the frictional force is equal to μR_N. Frictional force F' acts in the opposite direction of motion of the body. Till the motion just begins, the body is in equilibrium under the action of the following forces :

1. Applied force F
2. Force of friction F'
3. Weight of the body W and
4. Reaction R between body B and surface S.

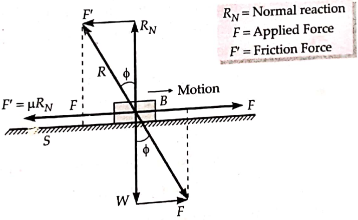


Fig. 3.2 Limiting Angle of Friction.

Actually, reaction R is equal and opposite to the resultant of F and W. It will be inclined at an angle ϕ with the normal reaction R_N. From the geometry of figure

$$\tan\phi = \frac{F}{W} = \frac{\mu R_N}{R_N} = \mu \qquad \text{or} \qquad \tan\phi = \mu$$

The angle ϕ is known as limiting angle of friction as it represents the maximum possible value of ϕ at the beginning of motion. It is the angle which the resultant reaction R makes with normal reaction R_N.

3.5 ANGLE OF REPOSE

A body B of weight W is resting on an inclined plane S as shown in Fig. 3.3. If the angle α of the inclined plane is such that the body B starts moving downwards on its own, then α is called the angle of repose or natural angle.

The weight of the body W can be resolved into *two* components :

(*i*) $W \sin \alpha$ parallel to the plane, and

(*ii*) $W \cos \alpha$ perpendicular to the plane

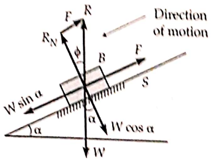


Fig. 3.3

The body begins to move downwards on the plane when

$$F = W \sin \alpha$$

where F is the friction force.

From the geometry of figure,

$$W \sin \alpha = F = \mu R_N = \mu W \cos \alpha$$

or $$\tan \alpha = \mu = \tan \phi$$

Thus $$\alpha = \phi$$

Hence, the angle of repose α is equal to the limiting angle of friction ϕ

3.6 MINIMUM FORCE REQUIRED TO DRAG A BODY ON ROUGH HORIZONTAL SURFACE

Suppose a body B of weight W is placed on rough horizontal plane S as shown in Fig. 3.4. An effort P is applied on the body subtending an angle θ with the horizontal. The minimum value of P is required to be found so that it just moves the body B on the horizontal surface S. Till equilibrium the forces acting on the body are :

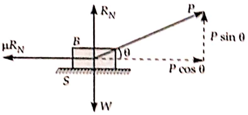


Fig. 3.4

1. Weight W
2. Effort P
3. Normal reaction R_N, and
4. Frictional force F

Now resolving the effort P into two components, one vertical and the other horizontal.

Vertical component $= P \sin \theta$

Horizontal component $= P \cos \theta$

Considering the vertical forces

$$R_N + P \sin \theta = W$$

$$R_N = W - P \sin \theta$$

Considering horizontal forces

$$P \cos \theta = F = \mu R_N \qquad ...(i)$$

Substituting the value of R_N in equation (*ii*) from equation (*i*)

$$P \cos\theta = \mu (W - P \sin\theta)$$

But we know that $\mu = \tan\phi$

So $$P \cos\theta = \tan\phi (W - P\sin\theta) = \frac{\sin\phi}{\cos\phi}(W - P\sin\theta)$$

$$P\cos\theta \, . \cos\phi = W\sin\phi - P\sin\theta \, . \sin\phi$$

$$P(\cos\theta \, . \cos\phi + \sin\theta \, . \sin\phi) = W\sin\phi$$

$$P\cos(\theta - \phi) = W\sin\phi$$

$$P = \frac{W\sin\phi}{\cos(\theta - \phi)}$$

The value of P is minimum when $\cos(\theta - \phi)$ is maximum.

So $$\cos(\theta - \phi) = 1 = \cos 0 \quad \text{or} \quad \theta - \phi = 0$$

$$\theta = \phi$$

Thus $$P_{min} = W\sin\phi = W\sin\theta$$

Hence, the effort P will be minimum if its angle of inclination θ with the horizontal is equal to the angle of friction ϕ.

3.7 BODY TENDING TO MOVE UPWARDS ON AN INCLINED PLANE

Suppose a body of weight W is lying on an inclined plane making an angle α with the horizontal as shown in Fig. 3.5. The effort P_0 is applied to move the body upwards and assuming no friction. P_0 makes an angle θ with the line of action of weight W.

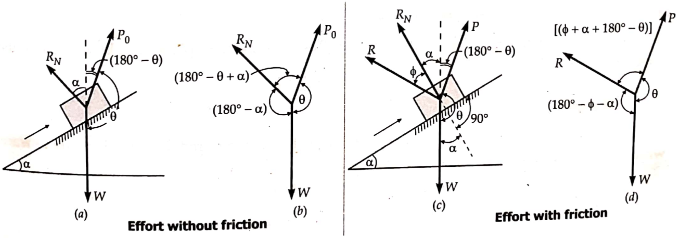


Fig. 3.5

To keep the body in equilibrium, the following forces act on it :

1. The weight W,
2. Effort P_0 and P without and with friction respectively
3. Normal reaction R_N
4. R is the resultant of R_N and frictional force

From law of forces [Fig. 3.5 (*b*)],

$$\frac{W}{\sin\{180^\circ - (\theta - \alpha)\}} = \frac{P_0}{\sin(180^\circ - \alpha)}$$

$$\frac{W}{\sin(\theta - \alpha)} = \frac{P_0}{\sin\alpha}$$

or

$$\frac{P_0}{W} = \frac{\sin\alpha}{\sin(\theta - \alpha)} \quad \text{...}(i)$$

Let us now assume that there is friction between the body and the plane. It is assumed that the reaction force R is inclined at an angle ϕ to normal reaction R_N where ϕ is the friction angle, as shown in Fig. 3.5 (*c*). Triangle of forces is shown in Fig. 3.5(*d*).

Applying Lami's theorem

$$\frac{P}{\sin(180^\circ - \phi - \alpha)} = \frac{W}{\sin\{180^\circ - (\theta - \alpha - \phi)\}}$$

or

$$\frac{P}{\sin(\phi + \alpha)} = \frac{W}{\sin(\theta - \alpha - \phi)}$$

$$\frac{P}{W} = \frac{\sin(\alpha + \phi)}{\sin(\theta - \alpha - \phi)} \quad \text{...}(ii)$$

Effort P will be minimum if $\sin(\theta - \alpha - \phi)$ is maximum. The maximum value of $\sin(\theta - \alpha - \phi)$ is 1.

So

$$\sin(\theta - \alpha - \phi) = 1 = \sin 90^\circ$$

or

$$\theta - \alpha - \phi = 90^\circ$$

or

$$\theta - (90^\circ + \alpha) = \phi$$

It means that the angle between the effort P and the inclined plane should be equal to the angle of friction. Thus $P_{min} = W\sin(\alpha + \phi)$

Efficiency

The efficiency of an inclined plane is the ratio of forces required to move the body upward without and with the consideration of friction.

Mathematically, $\eta = \frac{P_0}{P}$

Substituting the values of P_0 and P from equation (*i*) and (*ii*) in the above relation, we get

$$\eta = \frac{\dfrac{W\sin\alpha}{\sin(\theta - \alpha)}}{\dfrac{W\sin(\alpha + \phi)}{\sin(\theta - \alpha - \phi)}} = \frac{\sin\alpha \,.\, \sin(\theta - \alpha - \phi)}{\sin(\alpha + \phi)\,.\,\sin(\theta - \alpha)}$$

$$= \frac{\sin\alpha}{\sin(\alpha+\phi)} \cdot \frac{\sin\theta \cdot \cos(\alpha+\phi) - \cos\theta \cdot \sin(\alpha+\phi)}{\sin\theta \cdot \cos\alpha - \cos\theta \cdot \sin\alpha}$$

$$= \frac{\sin\alpha}{\sin(\alpha+\phi)} \cdot \frac{\sin\theta \cdot \sin(\alpha+\phi)\left[\dfrac{\cos(\alpha+\phi)}{\sin(\alpha+\phi)} - \dfrac{\cos\theta}{\sin\theta}\right]}{\sin\theta \cdot \sin\alpha\left[\dfrac{\cos\alpha}{\sin\alpha} - \dfrac{\cos\theta}{\sin\theta}\right]}$$

$$= \frac{\sin\alpha}{\sin(\alpha+\phi)} \cdot \frac{\sin(\alpha+\phi)}{\sin\alpha}\left[\frac{\cot(\alpha+\phi) - \cot\theta}{\cot\alpha - \cot\theta}\right]$$

$$\eta = \frac{\cot(\alpha+\phi) - \cot\theta}{\cot\alpha - \cot\theta}$$

NOTE *If $\theta = 90°$, applied effort is in horizontal direction, then the efficiency η is given by*

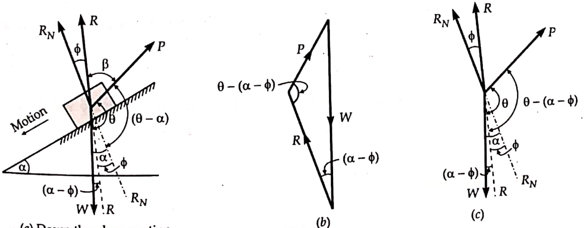
$$\eta = \frac{\cot(\alpha+\phi) - \cot 90°}{\cot\alpha - \cot 90°} = \frac{\cot(\alpha+\phi)}{\cot\alpha} = \frac{\tan\alpha}{\tan(\alpha+\phi)}$$

3.8 BODY MOVING DOWN THE PLANE

When friction is neglected, equation (*i*) of article 3.7

i.e.,
$$\frac{P_0}{W} = \frac{\sin\alpha}{\sin(\theta-\alpha)} \quad ...(i)$$

holds true for body moving down the plane also. Now let us take friction into consideration. The resultant reaction R is inclined by an angle ϕ to the normal reaction R_N towards right as shown in Fig. 3.6(*a*).



(*a*) Down the plane motion (*b*) (*c*)

Fig. 3.6

The angle between the line of action of forces P and R is $\theta - (\alpha - \phi)$ and between W and R is $(\alpha - \phi)$.

Scanned by CamScanner

Applying Lami's theorem to Figs. 3.6(*b* and *c*), we have

$$\frac{W}{\sin\{\theta-(\alpha-\phi)\}}=\frac{P}{\sin(\alpha-\phi)}$$

or $$P=W\frac{\sin(\alpha-\phi)}{\sin\{\theta-(\alpha-\phi)\}} \quad ...(ii)$$

NOTE (*i*) *When* $\theta=90°$, *P is applied horizontally, then equation (ii) can be written as*

$$P=\frac{W\sin(\alpha-\phi)}{\cos(\alpha-\phi)}=W\tan(\alpha-\phi)$$

(*ii*) *When* $\theta=90°+\alpha$, *P is parallel to the plane, then equation (ii) can be written as*

$$P=\frac{W\sin(\alpha-\phi)}{\sin\{(90°+\alpha)-(\alpha-\phi)\}}$$

$$=\frac{W\sin(\alpha-\phi)}{\sin(90°+\phi)}=\frac{W\sin(\alpha-\phi)}{\cos\phi}$$

$$=W\left[\frac{\sin\alpha\cos\phi}{\cos\phi}-\frac{\sin\phi\cos\alpha}{\cos\phi}\right]=W(\sin\alpha-\tan\phi\cos\alpha)$$

$$P=W(\sin\alpha-\mu\cos\alpha)$$

Efficiency of the inclined plane (Motion down the plane)

Since P is less than P_0, so

$$\eta=\frac{P}{P_0}=\frac{W\sin(\alpha-\phi)}{\sin\{\theta-(\alpha-\phi)\}}\times\frac{\sin(\theta-\alpha)}{W\sin\alpha}$$

(*Solving in a manner similar to article 3.7*)

$$\eta=\frac{\cot\alpha-\cot\theta}{\cot(\alpha-\phi)-\cot\theta}$$

When there is no friction, $\phi=0$, it means with no friction, the efficiency will be 100%.

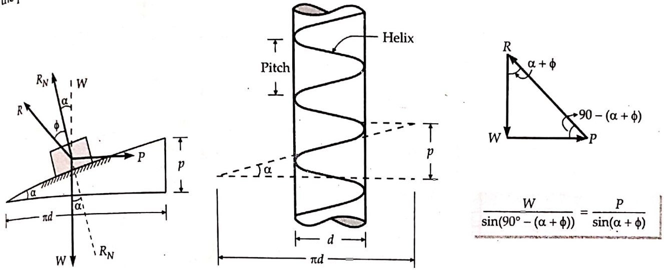
Theoretically, efficiency for the downward motion may be defined as the ratio of the forces on the body with and without friction.

3.9 SCREW AND NUT

A screw when developed is an inclined plane. Threads are cut on a cylindrical body of diameter d as shown in Fig. 3.7(*a*). The circumference of the cylinder is πd. The inclination of the plane 'α' is equal to the helix angle of the thread. The helix angle is given by

$$\tan\alpha=\frac{p}{\pi d},$$ where p is the pitch of the thread.

Pitch is the linear distance between two consecutive threads. The motion of the nut on a screw is analogous to the motion on an inclined plane as shown in Fig. 3.7(*b*). In this case effort P, required to move the body, acts horizontally. We have already discussed that it acts horizontally when $\theta = 90°$. When the nut comes down on screw, it is similar to the motion of body downward the plane. Certain expressions can be presented with the help of Figs. 3.7(*b*) and (*c*).



(*a*) (*b*) Motion of nut on screw (*c*) Force analysis of nut and screw

Fig. 3.7

Mechanical Advantage ($M.A.$)

$$= \frac{W}{P} = \frac{\cos(\alpha + \phi)}{\sin(\alpha + \phi)} = \cot(\alpha + \phi)$$

Velocity ratio ($V.R.$) $$= \frac{\text{Distance covered by } P}{\text{Distance covered by } W} = \frac{\pi d}{p} = \cot \alpha$$

Mechanical efficiency $$\eta_{Mech} = \frac{M.A.}{V.R.} = \frac{\cot(\alpha + \phi)}{\cot \alpha} = \frac{\tan \alpha}{\tan(\alpha + \phi)}$$

When the body or nut is lowered

$$P_0 = W \tan \alpha \quad \text{and} \quad P = W \tan(\alpha - \phi)$$

3.10 SCREW JACK WITH SQUARE THREADS

We have already discussed that motion of nut on the screw is analogous to sliding along an inclined plane.

Refer to Fig. 3.8

Let, P = Tangential force, r = mean radius of screw

W = axial load, α = inclination of thread

ϕ = friction angle

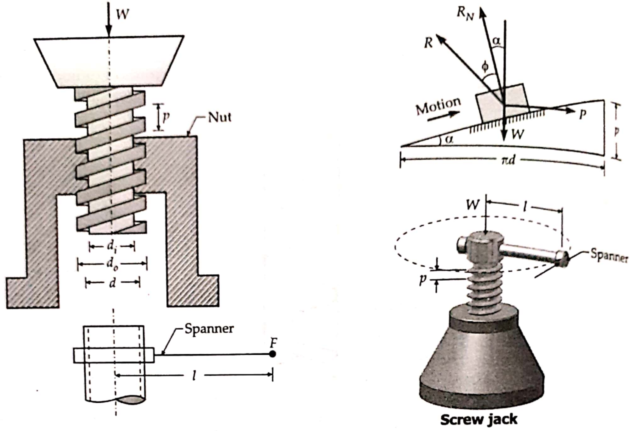


Fig. 3.8 Screw-Jack.

If the nut is rotated so that the screw moves against the axial load W, it is treated as motion upwards the inclined plane. In that case P and W are related as

$$P = W \frac{\sin(\alpha + \phi)}{\sin\{\theta - (\alpha + \phi)\}}$$

since $\theta = 90°$, so

$$P = W \frac{\sin(\alpha + \phi)}{\cos(\alpha + \phi)} = W \tan(\alpha + \phi) \quad ...(i)$$

The turning moment on the nut can be written as

$$T = P.r. = W \tan(\alpha + \phi) \times r \quad ...(ii)$$

If F is the effort applied to the spanner at a distance l from the axis of the screw, then

$$T = F.l \quad ...(iii)$$

or

$$F.l = W.r \tan(\alpha + \phi) \quad ...(iv)$$

In case the nut rotates in the opposite direction *i.e.*, load is to be lowered, the equation (iv) can be written as

$$T = -W.r \tan(\alpha - \phi) = W.r \tan(\phi - \alpha) \quad ...(v)$$

Efficiency of Screw-Jack

In article 3.9, the efficiency of nut and screw arrangement is given by

$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)}.$$

It is the efficiency for the upward motion.

Condition for Maximum Efficiency

We know that for η to be maximum

$$\frac{d\eta}{d\alpha} = 0$$

Thus
$$\frac{d\eta}{d\alpha} = 0 = \frac{\sec^2\alpha \tan(\alpha+\phi) - \sec^2(\alpha+\phi).\tan\alpha}{\tan^2(\alpha+\phi)}$$

$$\sec^2\alpha.\tan(\alpha+\phi) - \sec^2(\alpha+\phi).\tan\alpha = 0$$

or
$$\frac{1}{\cos^2\alpha}.\frac{\sin(\alpha+\phi)}{\cos(\alpha+\phi)} - \frac{1}{\cos^2(\alpha+\phi)}.\frac{\sin\alpha}{\cos\alpha} = 0$$

or
$$\frac{\sin(\alpha+\phi)}{\cos\alpha} - \frac{\sin\alpha}{\cos(\alpha+\phi)} = 0$$

$$\sin(\alpha+\phi).\cos(\alpha+\phi) - \sin\alpha.\cos\alpha = 0$$

$$2\sin(\alpha+\phi)\cos(\alpha+\phi) - 2\sin\alpha\cos\alpha = 0$$

$$\sin 2(\alpha+\phi) = \sin 2\alpha$$

or
$$2(\alpha+\phi) = \pi - 2\alpha \qquad [\because \sin\theta = \sin(\pi-\theta)]$$

$$\alpha = \frac{\pi}{4} - \frac{\phi}{2}$$

Again writing the expression for efficiency and substituting the value of $\alpha = \frac{\pi}{4} - \frac{\phi}{2}$

$$\eta_{max} = \frac{\tan\alpha}{\tan(\alpha+\phi)} = \frac{\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}{\tan\left(\frac{\pi}{4}-\frac{\phi}{2}+\phi\right)} = \frac{\tan\left(45^\circ-\frac{\phi}{2}\right)}{\tan\left(45^\circ+\frac{\phi}{2}\right)}$$

$$= \frac{\tan 45^\circ - \tan\frac{\phi}{2}}{1+\tan 45^\circ \tan\frac{\phi}{2}} . \frac{1-\tan 45^\circ.\tan\phi/2}{\tan 45^\circ + \tan\phi/2} = \frac{\left(1-\tan\frac{\phi}{2}\right)^2}{\left(1+\tan\frac{\phi}{2}\right)^2}$$

$$= \frac{\left(1-\frac{\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}}\right)^2}{\left(1+\frac{\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}}\right)^2} = \frac{\left(\cos\frac{\phi}{2}-\sin\frac{\phi}{2}\right)^2}{\left(\cos\frac{\phi}{2}+\sin\frac{\phi}{2}\right)^2} = \frac{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}-2\sin\frac{\phi}{2}.\cos\frac{\phi}{2}}{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}+2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$

$$\eta_{max} = \frac{1-\sin\phi}{1+\sin\phi} \qquad \left(\because \cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2} = 1 \text{ and } 2\sin\frac{\phi}{2}.\cos\frac{\phi}{2} = \sin\phi\right)$$

$$\text{Lead} = \text{number of starts} \times \text{pitch}$$

$$d = d_0 - \frac{p}{2} = d_i + \frac{p}{2}$$

where d = mean diameter of the screw

d_0 = outside diameter of the screw

d_i = inside diameter of the screw

NOTE *Generally, the axial load W is taken up by the thrust collar of mean radius R, then total torque required to overcome friction is given by*

$$T = P.r + \mu_1 W.R$$

where μ_1 is the coefficient of friction for the collar.

3.11 SCREW JACK WITH V-THREADS

In V-threads the axial load W does not act perpendicular to the surface of the threads. The axial component of the normal reaction R_N is kept equal to W, as shown in Fig. 3.9.

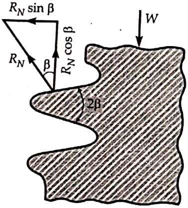


Fig. 3.9 V-thread.

2β = Angle of the V-thread

β = semi-angle of V-thread

$$R_N \cos\beta = W$$

$$R_N = \frac{W}{\cos\beta}$$

Frictional force tangential to the thread surface is given by

$$= \mu R_N = \mu W/\cos\beta = \mu_1.W$$

where $\mu_1 = \dfrac{\mu}{\cos\beta}$ and may be termed as virtual coefficient of friction. A given load may be lifted by applying lesser force by square threads as compared to V-threads. But V-threads are capable of taking more loads as compared to square threads.

3.12 OVER-HAULING AND SELF-LOCKING SCREWS

Refer this equation to lower the load, $P = W\tan(\phi - \alpha)$

If $\alpha > \phi$, the nut and the load placed on it will start moving downwards. It will be required to apply force to stop the downward motion. Such a state is termed as over-hauling of screws. This undesirable effect is removed by keeping the value of α always less than ϕ.

On the other hand, if $\phi > \alpha$ torque will be positive, so an effort will be required to lower the load. This type of screw is termed as self-locking screw. Thus for a screw to be self-locking friction angle ϕ must be greater than helix angle α.

Case II : Now consider the equilibrium of wedge B under the application of forces : (Refer to Fig. 3.14)

(*i*) Reaction force M from side wall,

(*ii*) Load R' applied on slider (wedge) downwards, and

(*iii*) The resultant reaction R_1 from wedge

Applying Lami's theorem,

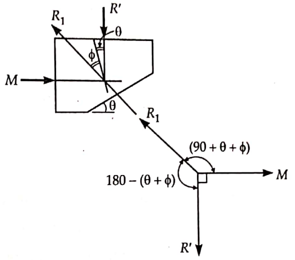


Fig. 3.14

$$\frac{R_1}{\sin 90°} = \frac{M}{\sin [180 - (\theta + \phi)]}$$

$$= \frac{R'}{\sin \{90 + \theta + \phi\}}$$

$$R_1 = \frac{M}{\sin (\theta + \phi)} = \frac{R'}{\cos (\theta + \phi)}$$

$$R' = R_1 \cos (\theta + \phi) = \frac{F}{\sin (\theta + \phi)} \cos (\theta + \phi)$$

$$R' = F \cot (\theta + \phi)$$

(substituting from equation (*i*) for R_1)

Efficiency of the system

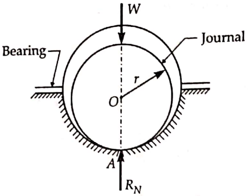
$$\eta = \frac{R'}{R} = \frac{F \cot (\theta + \phi)}{F \cot \theta}$$

$$\eta = \frac{\cot (\theta + \phi)}{\cot \theta} ; \quad \text{where } \phi = \tan^{-1} \mu$$

3.14 FRICTION IN TURNING PAIRS—FRICTION CIRCLE

When a shaft rotates in a bearing some power is lost due to friction between the shaft and bearing surface. When the shaft is at rest in the bearing as shown in Fig. 3.15(*a*), the weight W of the rotating element (called journal) acts at point A. The fixed element which is the bearing offers a normal reaction R_N upwards. The radius of the journal is kept less than that of the bearing due to running fit tolerances which vary with the function of journal bearing. Point A is known as the seat of the pressure of the bearing.

When shaft (journal) rotates, say clockwise, the point of contact A will be shifted to the right to point B as shown in Fig. 3.15(*b*). There will be two forces acting on the shaft at point B; the normal reaction R_N and the frictional force μR_N which act opposite to direction of rotation and tangential at B. The resultant reaction produced by the bearing will be R which is inclined at an angle ϕ with R_N.



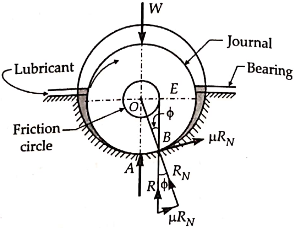


Fig. 3.15

Let ϕ = Angle between R and R_N $\quad$ T = Frictional torque

r = Radius of journal = OB

μ = Coefficient of friction between the journal & bearing.

Since there is no other force, so

$$W = R$$

Frictional torque can be written as

$$T = W.OE = W.OB \sin\phi$$

$$= W.r \sin\phi = W.r \tan\phi \quad (\because \sin\phi = \tan\phi \text{ when } \phi \text{ is very small})$$

$$= W.r.\mu \quad (\because \mu = \tan\phi)$$

$$T = \mu W r$$

If a circle is drawn with O as centre and OE as radius, it is called a friction circle.

Power lost in friction is given by

$$P = T.\omega \text{ watt}$$

where ω is the angular speed of shaft.

3.15 PIVOT AND COLLAR FRICTION

The rotating shafts are quite frequently subjected to axial load which is known as thrust. This axial load produces lateral motion of the shaft along its axis which is not desirable. In order to prevent the lateral motion of shaft one or more bearing surfaces called pivots and collars are provided. These surfaces may be flat or conical. A bearing surface provided at the end of a shaft is known as pivot, and a collar is provided along with the length of shaft with bearing surface of revolution as shown in Fig. 3.16. The collars and pivots take the axial load of the shaft. For example, the shafts of steam turbines, propeller shafts of ships etc. exert an axial thrust on pivot or collars. When the bearing is at the end of vertical shaft, it is called foot step bearing. There is some friction between the shaft and the bearing which leads to some loss of power.

6.8. PIVOT AND COLLAR BEARING

The rotating shafts are frequently subjected to axial thrust. These shafts can be kept in correct axial position if bearing surfaces are provided. The bearing surfaces which are flat or conical carry the axial thrust. The bearing surfaces placed at the end of a shaft are known as **pivots**. The pivot may have a flat surface or a conical surface or truncated conical surface as shown in Fig. 6.8 (*a*), (*b*) and (*c*) respectively.

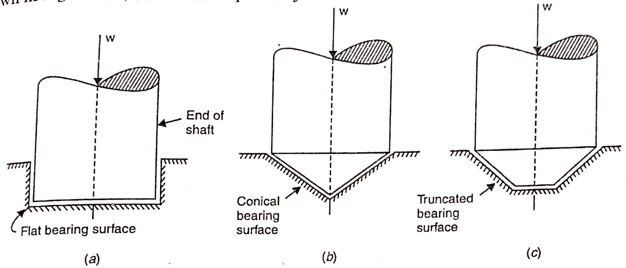


Fig. 6.8

The bearing surfaces provided at any position along the shaft (but not at the end of the shaft), to carry the axial thrust, is known as **collar**. The surface of the collar may be flat (normal to the axis of shaft) or of conical shape as shown in Fig. 6.9 (*a*) and (*b*) respectively. The collar bearings are also known as *thrust bearings*.

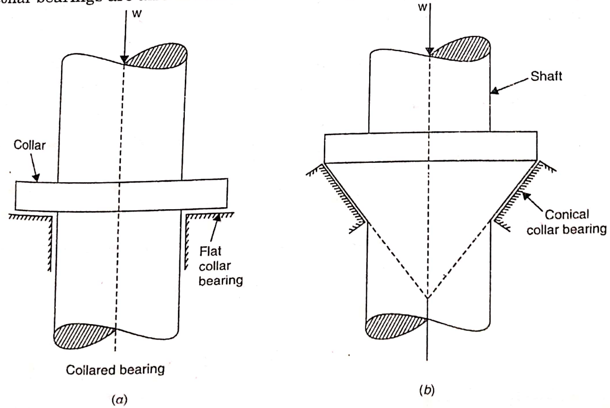


Fig. 6.9

For a new bearing, the contact between the shaft and bearing may be good over the whole surface. This means that the pressure over the rubbing surfaces may be assumed as uniformly distributed. But when the bearing becomes old, all parts of the rubbing surfaces will not move with the same velocity and hence the wear will be different at different radii. The pressure distribution will not be uniform. The rate of wear of surfaces depends upon the pressure and the rubbing velocities between the surfaces.

The design of bearings is based on the following assumptions though neither of them is strictly true :

(*i*) the pressure is uniformly distributed over the bearing surfaces, and

(*ii*) the wear is uniform over the bearing surface.

The power lost, due to friction in pivot and collar bearings, are calculated on the above two assumptions.

6.9. FLAT PIVOT

The bearing surface placed at the end of the shaft is known as pivot. If the surface is flat as shown in Fig. 6.10, then the bearing surface is called flat pivot or foot-step. There will be friction along the surface of contact between the shaft and bearing. The power lost can be obtained by calculating the torque.

Let W = Axial load, or load transmitted to the bearing surface,

R = Radius of pivot,

μ = Co-efficient of friction,

p = Intensity of pressure in N/m^2, and

T = Total frictional torque.

Consider a circular ring of radius r and thickness dr as shown in Fig. 6.10.

∴ Area of ring $= 2\pi r.dr$

We will consider the two cases, namely

(*i*) case of uniform pressure over bearing surface and

(*ii*) case of uniform wear over bearing surface.

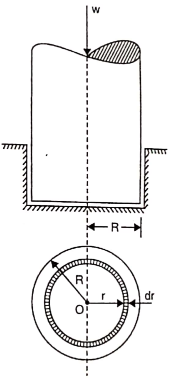


Fig. 6.10

6.9.1. Case of Uniform Pressure. When the pressure is assumed to be uniform over the bearing surface, then intensity of pressure (p) is given by

$$p = \frac{\text{Axial load}}{\text{Area of cross-section}} = \frac{W}{\pi R^2} \quad ...(i)$$

Now let us find the load transmitted to the ring and also frictional torque on the ring.

Load transmitted to the ring,

$$dW = \text{Pressure on the ring} \times \text{Area of ring}$$
$$= p \times 2\pi r dr$$

Frictional force* on the ring,

$$dW = \mu \times \text{load on ring}$$

*Load on the ring is vertically downward. Hence frictional force on the ring will be equal to $\mu \times$ normal reaction *i.e.*, $\mu \times$ load on ring. Here normal reaction on the ring is equal to load on the ring. Hence frictional force on ring, $dF = \mu \times dW$.

$$= \mu \times p \times 2\pi r dr$$

∴ Frictional torque on the ring

$$= \text{Friction force} \times \text{Radius of ring} = dF \times r$$

∴ Frictional torque, $dT = \mu \times p \times 2\pi r dr \times r$

$$= 2\pi\mu p r^2 dr$$

The total frictional torque (T) will be obtained by integrating the above equation from 0 to R.

∴ Total frictional torque, $T = \int_0^R 2\pi\mu p r^2 \, dr$

$$= 2\pi\mu p \int_0^R r^2 dr \qquad (\because \ \mu \text{ and } p \text{ are constant})$$

$$= 2\pi\mu p \left[\frac{r^3}{3}\right]_0^R = 2\pi\mu p \left[\frac{R^3}{3}\right] = \frac{2}{3}\pi\mu p R^3$$

$$= \frac{2}{3}\pi \times \mu \times \frac{W}{\pi R^2} \times R^3 \qquad \left(\because \text{ From } (i),\ p = \frac{W}{\pi R^2}\right)$$

$$= \frac{2}{3}\mu W R \qquad \ldots(6.2)$$

∴ Power lost in friction $= T \times \omega$

$$= T \times \frac{2\pi N}{60} \qquad \left(\because \ \omega = \frac{2\pi N}{60}\right)$$

$$= \frac{2\pi N T}{60} \qquad \ldots(6.3)$$

6.9.2. Case of Uniform Wear. For the uniform wear of the bearing surface, the load transmitted to the various circular rings should be same (or should be constant). But load transmitted to any circular ring is equal to the product of pressure and area of the ring. Hence for uniform wear, the product of pressure and area of ring should be constant. Area of the ring is directly proportional to the radius of the ring. Hence for uniform wear, the product of pressure and radius should be constant or $p \times r$ = constant.

Hence for uniform wear, we have

$$p \times r = \text{Constant} \quad (\text{say } C) \qquad \ldots(6.4)$$

$$\therefore \quad p = \frac{C}{r} \qquad \ldots(i)$$

Now we know that load transmitted to the ring

$$= \text{Pressure} \times \text{Area or ring}$$

$$= p \times 2\pi r dr$$

$$= \frac{C}{r} \times 2\pi r dr \qquad \left[\because \text{ From } (i),\ p = \frac{C}{r}\right]$$

$$= 2\pi C \, dr \qquad \ldots(ii)$$

Total load transmitted to the bearing, is obtained by integrating the above equation from 0 to R.

∴ Total load transmitted to the bearing

$$= \int_0^R 2\pi C dr = 2\pi C \int_0^R dr = 2\pi C \Big[r\Big]_0^R$$

$$= 2\pi C R$$

But total load transmitted to the bearing = W

$$\therefore \quad 2\pi CR = W$$

or

$$C = \frac{W}{2\pi R} \quad \text{...(iii)}$$

Now frictional force on the ring,

$$dF = \mu \times \text{Load on ring} = \mu \times dW$$
$$= \mu \times 2\pi C dr \quad [\because \text{ From (ii), load on ring} = 2\pi C dr]$$

Hence frictional torque on the ring,

$$dT = \text{Frictional force on ring} \times \text{radius}$$
$$= \mu \times 2\pi C dr \times r$$

$\therefore$ Total frictional torque, $T = \int_0^R dT$

$$= \int_0^R \mu \times 2\pi Cr\, dr$$
$$= 2\pi\mu C \int_0^R r\, dr \quad [\mu \text{ and } C \text{ are constant}]$$
$$= 2\pi\mu C \left[\frac{r^2}{2}\right]_0^R = 2\pi\mu C \times \frac{R^2}{2}$$
$$= 2\pi\mu \times \frac{W}{2\pi R} \times \frac{R^2}{2} \quad \left[\because \text{ From (iii), } C = \frac{W}{2\pi R}\right]$$

or

$$T = \frac{1}{2} \times \mu WR \quad \text{...(6.5)}$$

$\therefore$ Power lost in friction $= T \times \omega = \dfrac{2\pi NT}{60}$.

Problem 6.2. *Find the power lost in friction assuming (i) uniform pressure and (ii) uniform wear when a vertical shaft of 100 mm diameter rotating at 150 r.p.m. rests on a flat end foot step bearing. The co-efficient of friction is equal to 0.05 and shaft carries a vertical load of 15 kN.*

Sol. Given :

Diameter, $D = 100$ mm $= 0.1$ m $\quad \therefore \quad R = \dfrac{0.1}{2} = 0.05$ m

Speed, $N = 150$ r.p.m., Friction co-efficient, $\mu = 0.05$

Load, $W = 15$ kN $= 15 \times 10^3$ N

(i) Power lost in friction assuming uniform pressure

For uniform pressure, frictional torque is given by equation (6.2) as

$$T = \frac{2}{3}\mu WR$$
$$= \frac{2}{3} \times 0.05 \times 15 \times 10^3 \times 0.05 \text{ Nm} = 25 \text{ Nm}$$

Power lost in friction $= \dfrac{2\pi NT}{60}$

$$= \frac{2\pi \times 150 \times 25}{60} \text{ W} = \mathbf{392.7 \text{ W}}. \quad \textbf{Ans.}$$

(ii) *Power lost in friction assuming uniform wear*

For uniform wear, the frictional torque is given by equation (6.5) as

$$T = \frac{1}{2}\mu WR$$

$$= \frac{1}{2} \times 0.05 \times 15 \times 10^3 \times 0.05 \text{ Nm} = 18.75 \text{ Nm}$$

∴ Power lost in friction $= \dfrac{2\pi NT}{60}$

$$= \frac{2\pi \times 150 \times 18.75}{60} \text{ W} = \mathbf{294.5 \text{ W}}. \quad \textbf{Ans.}$$

6.10. CONICAL PIVOT

The bearing surface placed at the end of a shaft and having a conical surface, is known as conical pivot as shown in Fig. 6.11.

Let W = Axial load, or load transmitted to the bearing surface

μ = Co-efficient of friction

R = Radius of shaft

α = Semi-angle of the cone

p = Pressure intensity normal to the cone surface.

Consider a circular ring of radius r and thickness dr. The actual thickness of the sloping ring will be $\dfrac{dr}{\sin\alpha}$ as shown in Fig. 6.11 (b) in which $AB = dr$ on enlarged scale, angle $ACB = \alpha$ and sloping length of ring $= AC = \dfrac{AB}{\sin\alpha} = \dfrac{dr}{\sin\alpha}$.

∴ Area of ring along conical surface

$= 2\pi r \times$ Actual thickness of sloping ring

$$= 2\pi r \times \frac{dr}{\sin\alpha}$$

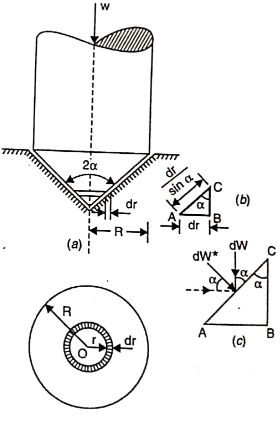


Fig. 6.11

Now we will consider two cases namely :

(i) Case of uniform pressure

(ii) Case of uniform wear.

6.10.1. Case of Uniform Pressure. Let us first find the load acting on the circular ring, normal to the conical surface.

∴ Load on the ring normal to conical surface,

$$dW^* = \text{Pressure} \times \text{Area of ring along conical surface}$$

$$= p \times 2\pi r \times \frac{dr}{\sin\alpha}$$

Vertical component of the above load [Refer to Fig. 6.11 (c)]

$$dW = \left[p \times 2\pi r \times \frac{dr}{\sin \alpha}\right] \times \sin \alpha \qquad (\because \; dW = dW^* \sin \alpha)$$

$$= p \times 2\pi r \times dr$$

∴ Total vertical load transmitted to the bearing

$$= \int_0^R p \times 2\pi r \times dr \qquad ...(6.6)$$

$$= p \times 2\pi \int_0^R r\, dr \quad (\because \text{ pressure is uniform and hence constant})$$

$$= p \times 2\pi \times \left[\frac{r^2}{2}\right]_0^R = p \times 2\pi \times \frac{R^2}{2} = p \times \pi R^2$$

But total vertical load transmitted is also = W

∴

$$W = p \times \pi R^2 \qquad ...(i)$$

Also

$$p = \frac{W}{\pi R^2} \qquad ...(ii)$$

The above equation shows that pressure intensity is independent of the angle of the cone.

Now the frictional force on the ring along the conical surface,

$$dF = \mu \times \text{Load on ring normal to conical surface} = \mu \times dW^*$$

$$= \mu \times \left(p \times 2\pi r \times \frac{dr}{\sin \alpha}\right)$$

∴ Moment of this frictional force about the shaft axis (dT)

= Frictional torque on the ring

= Frictional force × Radius = $dF \times r$

$$= \mu \times \left(p \times 2\pi r \times \frac{dr}{\sin \alpha}\right) \times r \qquad ...(6.7)$$

Total moment of the frictional force about the shaft axis or total frictional torque on the conical surface is obtained by integrating the above equation from 0 to R.

∴ Total frictional torque,

$$T = \int_0^R \mu \times p \times 2\pi r \times \frac{dr}{\sin \alpha} \times r$$

$$= \frac{2\pi \times \mu \times p}{\sin \alpha} \int_0^R r^2\, dr \qquad (\because \; \mu, p \text{ and } \alpha \text{ are constant})$$

$$= \frac{2\pi \times \mu \times p}{\sin \alpha}\left[\frac{r^3}{3}\right]_0^R = \frac{2\pi \times \mu \times p}{\sin \alpha} \times \frac{R^3}{3}$$

$$= \frac{2\pi \times \mu}{\sin \alpha} \times \frac{W}{\pi R^2} \times \frac{R^3}{3} \qquad \left[\because \text{ From } (ii),\; p = \frac{W}{\pi R^2}\right]$$

$$= \frac{2}{3} \times \frac{\mu W R}{\sin \alpha} \qquad ...(6.8)$$

∴ Power lost in friction $= \dfrac{2\pi N T}{60}$.

6.10.2. Case of Uniform Wear. From equation (6.4), for uniform wear, we know that

$$p \times r = \text{Constant} \quad (\text{say} = C)$$

$$\therefore \quad p \times r = C$$

$$\text{or} \quad p = \frac{C}{r}$$

From equation (6.6), total vertical load transmitted to the bearing

$$= \int_0^R p \times 2\pi r \times dr$$

$$= \int_0^R \frac{C}{r} \times 2\pi r \times dr \qquad \left(\because \; p = \frac{C}{r}\right)$$

$$= 2\pi \times C \int_0^R dr = 2\pi \times C \Big[r\Big]_0^R$$

$$= 2\pi \times C \times R$$

But total vertical load transmitted to the bearing is also equal to W

$$\therefore \quad W = 2\pi \times C \times R$$

$$\text{or} \quad C = \frac{W}{2\pi R}$$

Now the frictional torque on the ring is given by the equation (6.7) as

$$dT = \mu \times p \times 2\pi r \times \frac{dr}{\sin \alpha} \times r$$

$$= \mu \times \frac{C}{r} \times 2\pi r \times \frac{dr}{\sin \alpha} \times r \qquad \left(\because \; p = \frac{C}{r} \text{ for uniform wear}\right)$$

$$= 2\pi\mu C \times r \times \frac{dr}{\sin \alpha} \qquad \text{...(6.8A)}$$

$$= 2\pi\mu \times \frac{W}{2\pi R} \times r \times \frac{dr}{\sin \alpha} \qquad \left(\because \; C = \frac{W}{2\pi R}\right)$$

$\therefore$ Total frictional torque,

$$T = \int_0^R dT = \int_0^R 2\pi\mu \times \frac{W}{2\pi R} \times r \times \frac{dr}{\sin \alpha}$$

$$= 2\pi\mu \times \frac{W}{2\pi R} \times \frac{1}{\sin \alpha} \int_0^R r\, dr = 2\pi\mu \times \frac{W}{2\pi R} \times \frac{1}{\sin \alpha} \times \frac{R^2}{2}$$

$$= \frac{1}{2} \times \frac{\mu W R}{\sin \alpha} \qquad \text{...(6.9)}$$

$$\therefore \quad \text{Power lost in friction} = \frac{2\pi N T}{60}.$$

6.10.3. Truncated Conical Pivot. Fig. 6.12 shows the truncated conical pivot of external and internal radii as r_1 and r_2.

***(i)* Case of Uniform Pressure**

Total vertical load transmitted to the bearing is obtained from equation (6.6) in which the limits of integration are from r_2 to r_1.

∴ Total vertical load transmitted to the bearing

$$= \int_{r_2}^{r_1} p \times 2\pi r \times dr$$

$$= p \times 2\pi \int_{r_2}^{r_1} r \times dr$$

(p is constant for uniform pressure)

$$= p \times 2\pi \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = p \times 2\pi \left[\frac{r_1^2 - r_2^2}{2}\right]$$

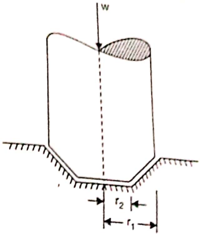


Fig. 6.12

But total vertical load $= W$

$$\therefore \quad W = p \times 2\pi \left[\frac{r_1^2 - r_2^2}{2}\right] = p \times \pi\,[r_1^2 - r_2^2]$$

or

$$p = \frac{W}{\pi(r_1^2 - r_2^2)} \qquad \text{...(6.9 A)}$$

Total frictional torque on the truncated conical surface is obtained by integrating equation (6.7) from r_2 to r_1.

$$\therefore \quad T = \int_{r_2}^{r_1} \mu \times p \times 2\pi r \times \frac{dr}{\sin\alpha} \times r$$

$$= \frac{2\mu \times \pi \times p}{\sin\alpha} \int_{r_2}^{r_1} r^2\, dr \qquad (\mu, p \text{ and } \alpha \text{ are constant})$$

$$= \frac{2\pi \times \mu \times p}{\sin\alpha} \left[\frac{r^3}{3}\right]_{r_2}^{r_1} \frac{2\pi \times \mu \times p}{\sin\alpha} \left[\frac{r_1^3 - r_2^3}{3}\right]$$

$$= \frac{2\pi \times \mu}{\sin\alpha} \times \frac{W}{\pi(r_1^2 - r_2^2)} \times \left[\frac{r_1^3 - r_2^3}{3}\right] \qquad \left[\because \text{ From (6.9A), } p = \frac{W}{\pi(r_1^2 - r_2^2)}\right]$$

$$= \frac{2}{3}\frac{\mu W}{\sin\alpha} \times \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) \qquad \text{...(6.10)}$$

(*ii*) Case of Uniform Wear

For uniform wear, $p \times r = C$

or

$$p = \frac{C}{r}$$

The total vertical load transmitted to the bearing is obtained from equation (6.6) in which limits of integration are from r_2 to r_1.

∴ Total vertical load transmitted

$$= \int_{r_2}^{r_1} p \times 2\pi r \times dr$$

$$= \int_{r_2}^{r_1} \frac{C}{r} \times 2\pi r \times dr \qquad \left(\because \ p = \frac{C}{r}\right)$$

$$= 2\pi C \int_{r_2}^{r_1} dr = 2\pi C \left[r\right]_{r_2}^{r_1} = 2\pi C\,[r_1 - r_2]$$

But total vertical load $= W$

$\therefore$
$$W = 2\pi C\,[r_1 - r_2]$$

or
$$C = \frac{W}{2\pi[r_1 - r_2]}$$

The total frictional torque for uniform wear is obtained by integrating the equation (6.8A) from r_2 to r_1.

$\therefore$ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu \times C \times r \times \frac{dr}{\sin\alpha}$$

$$= \int_{r_2}^{r_1} 2\pi\mu \times \frac{W}{2\pi(r_1 - r_2)} \times r \times \frac{dr}{\sin\alpha} \qquad \left(\because\ C = \frac{W}{2\pi(r_1 - r_2)}\right)$$

or
$$T = \frac{2\pi\mu \times W}{2\pi(r_1 - r_2)} \times \frac{1}{\sin\alpha}\int_{r_2}^{r_1} r\,dr$$

$$= \frac{\mu W}{(r_1 - r_2)} \times \frac{1}{\sin\alpha}\left[\frac{r^2}{2}\right]_{r_2}^{r_1} \frac{\mu W}{(r_1 - r_2)} \times \frac{1}{\sin\alpha}\left[\frac{r_1^2 - r_2^2}{2}\right]$$

$$= \frac{1}{2} \times \frac{\mu W}{\sin\alpha}(r_1 + r_2) \qquad \text{...(6.11)}$$

Problem 6.3. *A conical pivot with angle of cone as 120°, supports a vertical shaft of diameter 300 mm. It is subjected to a load of 20 kN. The co-efficient of friction is 0.05 and the speed of shaft is 210 r.p.m. Calculate the power lost in friction assuming (i) uniform pressure and (ii) uniform wear.*

Sol. Given :

$2\alpha = 120°$ $\quad\therefore\quad \alpha = 60°$;
$D = 300$ mm $= 0.3$ m $\quad\therefore\quad R = 0.15$ m ;
$W = 20$ kN $= 20 \times 10^3$ N ; $\quad \mu = 0.05$;
$N = 210$ r.p.m.

(i) Power lost in friction for uniform pressure

The frictional torque is given by equation (6.8) as

$$T = \frac{2}{3} \times \frac{\mu WR}{\sin\alpha}$$

$$= \frac{2}{3} \times \frac{0.05 \times 20 \times 10^3 \times 0.15}{\sin 60°} = 115.53 \text{ Nm}$$

$\therefore$ Power lost
$$= \frac{2\pi NT}{60}$$

$$= \frac{2\pi \times 210 \times 115.53}{60} = 2540.6 \text{ W} = \mathbf{2.54\ kW.} \quad \textbf{Ans.}$$

(ii) Power lost in friction for uniform wear

The friction torque is given by equation (6.9) as

$$T = \frac{1}{2} \times \frac{\mu WR}{\sin\alpha}$$

$$= \frac{1}{2} \times \frac{0.05 \times 20 \times 10^3 \times 0.15}{\sin 60°} = 86.60 \text{ Nm}$$

$\therefore$ Power lost

$$= \frac{2\pi NT}{60}$$

$$= \frac{2\pi \times 210 \times 86.6}{60} = 1904.4 \text{ W} = \mathbf{1.9044 \text{ kW}}. \quad \textbf{Ans.}$$

Problem 6.4. *A load of 25 kN is supported by a conical pivot with angle of cone as 120°. The intensity of pressure is not to exceed 350 kN/m². The external radius is 2 times the internal radius. The shaft is rotating at 180 r.p.m. and co-efficient of friction is 0.05. Find the power absorbed in friction assuming uniform pressure.*

Sol. Given :

Load, $W = 25$ kN $= 25 \times 10^3$ N ; Angle of cone, $2\alpha = 120°$ or $\alpha = 60°$

Pressure, $p = 350$ kN/m² $= 350 \times 10^3$ N/m² ; External radius $= 2 \times$ internal radius

Hence $r_1 = 2r_2$; Speed, $N = 180$ r.p.m. ; and $\mu = 0.05$

Using equation (6.9A) for uniform pressure, we get

$$p = \frac{W}{\pi(r_1^2 - r_2^2)}$$

or
$$350 \times 10^3 = \frac{25 \times 10^3}{\pi[(2r_2)^2 - r_2^2]} \qquad (\because \; r_1 = 2r_2)$$

or
$$[(2r_2)^2 - r_2^2] = \frac{25}{\pi \times 350}$$

$$= 0.02273$$

or
$$3r_2^2 = 0.02273$$

or
$$r_2 = \sqrt{\frac{0.02273}{3}} = 0.087 \text{ m}$$

$\therefore$
$$r_1 = 2r_2 = 2 \times 0.087 = 0.174 \text{ m}$$

To find the power absorbed in friction, first calculate the total frictional torque when pressure is uniform.

Frictional torque when pressure is uniform is given by equation (6.10) as

$$T = \frac{2}{3} \times \frac{\mu W}{\sin \alpha} \times \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$$

$$= \frac{2}{3} \times \frac{0.05 \times 25 \times 10^3}{\sin 60°} \left(\frac{0.174^3 - 0.087^3}{0.174^2 - 0.087^2}\right)$$

$$= 962.278 \left(\frac{0.005268 - 0.0006585}{0.03027 - 0.007569}\right) = 962.278 \left(\frac{0.0046095}{0.0227}\right)$$

$$= 195.37 \text{ Nm}$$

$\therefore$ Power absorbed in friction,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 180 \times 195.37}{60} = 3682.6 \text{ W} = \mathbf{3.6826 \text{ kW}}. \quad \textbf{Ans.}$$

6.10.4. Flat Collar. The bearing surface provided at any position along the shaft (but not at the end of the shaft), to carry axial thrust is known as collar which may be flat or conical. If the surface is flat, then bearing surface is known as flat collar as shown in Fig. 6.13. The collar bearings are also known as *thrust bearings*. The power lost in friction can be obtained by calculating the torque.

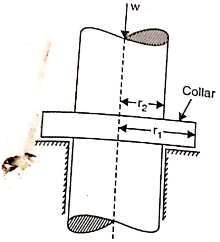


Fig. 6.13

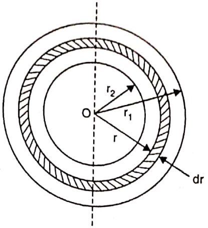


Fig. 6.13 (*a*)

Let r_1 = External radius of collar

r_2 = Internal radius of collar

p = intensity of pressure

W = Axial load or total load transmitted to the bearing surface

μ = Co-efficient of friction

T = Total frictional torque

Consider a circular ring of radius r and thickness dr as shown in Fig. 6.13 (*a*).

∴ Area of ring $= 2\pi r\, dr$.

Load on the ring = Pressure × Area of ring

$= p \times 2\pi r\, dr$ ...(*i*)

Frictional force on the ring = μ × Load on ring

$= \mu \times p \times 2\pi r\, dr$

Frictional torque on the ring, dT = Frictional force × Radius

$= (\mu \times p \times 2\pi r\, dr) \times r$

$= 2\pi\mu p r^2\, dr$

∴ Total frictional torque,

$$T = \int_{r_2}^{r_1} dT$$

$$= \int_{r_2}^{r_1} 2\pi\mu p r^2 dr \quad ...(ii)$$

(*i*) Uniform Pressure

p = Constant

Total load transmitted to the bearing

$$= \int_{r_2}^{r_1} \text{Load on ring}$$

or $$W = \int_{r_2}^{r_1} p \times 2\pi r\, dr \qquad (\because \text{ Load on ring from } (i) = p \times 2\pi r\, dr)$$

$$= p \times 2\pi \int_{r_2}^{r_1} r\, dr \qquad (\because \; p \text{ is constant})$$

$$= 2\pi p \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi p \left[\frac{r_1^2 - r_2^2}{2}\right] = \pi \times p\,[r_1^2 - r_2^2]$$

$\therefore$ $$p = \frac{W}{\pi[r_1^2 - r_2^2]} \qquad \text{...(6.12)}$$

Total frictional torque is given by equation (*ii*),

$\therefore$ $$T = \int_{r_2}^{r_1} 2\pi\mu p r^2\, dr$$

$$= 2\pi\mu p \int_{r_2}^{r_1} r^2\, dr \qquad (\because \; p \text{ is constant})$$

$$= 2\pi\mu p \left[\frac{r^3}{3}\right]_{r_2}^{r_1} \qquad \left[\text{But from (6.12), } p = \frac{W}{\pi(r_1^2 - r_2^2)}\right]$$

$$= 2\pi\mu p \left[\frac{r_1^3 - r_2^3}{3}\right] = 2\pi\mu \times \frac{W}{\pi(r_1^2 - r_2^2)} \times \left(\frac{r_1^3 - r_2^3}{3}\right)$$

$$= \frac{2}{3}\mu W \left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right] \qquad \text{...(6.13)}$$

$\therefore$ Power lost in friction,

$$P = \frac{2\pi NT}{60}$$

(*ii*) Uniform Wear

$$p \times r = \text{constant} \qquad (\text{say } C)$$

$\therefore$ $$p = \frac{C}{r}$$

Total load transmitted to the bearing

$$= \int_{r_2}^{r_1} \text{Load on ring} = \int_{r_2}^{r_1} p \times 2\pi r\, dr$$

$\therefore$ $$W = \int_{r_2}^{r_1} p \times 2\pi r\, dr$$

$$= \int_{r_2}^{r_1} \frac{C}{r} \times 2\pi r\, dr \qquad \left(\because \; p = \frac{C}{r}\right)$$

$$= 2\,\pi\, C \int_{r_2}^{r_1} dr \qquad (\because \; C \text{ is constant})$$

$$= 2\pi C \left[\, r \,\right]_{r_2}^{r_1} = 2\pi C\,[r_1 - r_2]$$

$$C = \frac{W}{2\pi(r_1 - r_2)} \qquad \text{...}(iii)$$

Total frictional torque is given by equation (*ii*),

$$T = \int_{r_2}^{r_1} 2\pi\mu p r^2\, dr$$

$$= 2\pi\mu \int_{r_2}^{r_1} p r^2\, dr \qquad \left(\text{Here } p \text{ is not constant it is } = \frac{C}{r}\right)$$

$$= 2\pi\mu \int_{r_2}^{r_1} \frac{C}{r}\, r^2\, dr$$

$$= 2\pi\mu \int_{r_2}^{r_1} Cr\, dr = 2\pi\mu C \int_{r_2}^{r_1} r\, dr \qquad (C \text{ is constant})$$

$$= 2\pi\mu C \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu C \left[\frac{r_2^2 - r_1^2}{2}\right]$$

$$= 2\pi\mu \times \frac{W}{2\pi(r_1 - r_2)} \times \left[\frac{r_2^2 - r_1^2}{2}\right] \qquad \left[\because C = \frac{W}{2\pi(r_1 - r_2)} \text{ from } (iii)\right]$$

$$= \frac{\mu W}{2}(r_1 + r_2) \qquad \ldots(6.14)$$

$\therefore$ Power lost in friction

$$P = \frac{2\pi NT}{60}$$

If the axial load on the bearing is too great, then the bearing pressure on the collar will become more than the limiting bearing pressure which is approximately equal to 400 kN/m². Hence to reduce the intensity of pressure on collar, two or more collars are used (or multi-collars are used) as shown in Fig. 6.14.

If n = number of collars in multi-collar bearing, then

(*i*) $n = \dfrac{\text{Total load}}{\text{Load permissible on one collar}}$

(*ii*) p = Intensity of the uniform pressure

$$= \frac{\text{Load}}{\text{No. of collars} \times \text{Area of one-collar}}$$

$$= \frac{W}{n \times \pi [r_1^2 - r_2^2]}$$

(*iii*) Total torque transmitted remains constant *i.e.*,

$$T = \frac{2}{3} \times \mu W \left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]$$

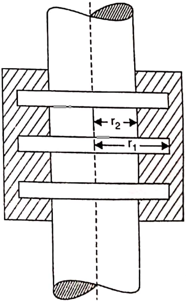


Fig. 6.14

Note. The frictional torque for uniform pressure is greater than that of uniform wear. Hence for safe design of bearing surfaces when power lost in friction is to be determined and no assumption is mentioned, assume uniform pressure. But when power transmitted is to be determined and no assumption is stated, assume uniform wear.

Problem 6.5. *In a collar thrust bearing the external and internal radii are 250 mm and 150 mm respectively. The total axial load is 50 kN and shaft is rotating at 150 r.p.m. The co-efficient of friction is equal to 0.05. Find the power lost in friction assuming uniform pressure.*

Sol. Given :

External radius, $r_1 = 250$ mm $= 0.25$ m

Internal radius, $r_2 = 150$ mm $= 0.15$ m

Total axial load, $W = 50$ kN $= 50 \times 10^3$ N

Speed, $N = 150$ r.p.m.

Co-efficient of friction, $\mu = 0.05$

For uniform pressure, the total frictional torque is given by equation (6.13), as

$$T = \frac{2}{3}\mu W\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]$$

$$= \frac{2}{3} \times 0.05 \times 50 \times 10^3 \left[\frac{0.25^3 - 0.15^3}{0.25^2 - 0.15^2}\right]$$

$$= 1666.67 \times \left(\frac{0.015625 - 0.003375}{0.0625 - 0.0225}\right) = 1666.67 \times \frac{0.01225}{0.04}$$

$$= 510.42 \text{ Nm}$$

$\therefore$ Power lost in friction, $$P = \frac{2\pi NT}{60}$$

$$= \frac{2\pi \times 150 \times 510.42}{60} = 8017.6 \text{ W} = \mathbf{8.0176\ kW.}\ \ \textbf{Ans.}$$

Problem 6.6. *In a thrust bearing the external and internal radii of the contact surfaces are 210 mm and 160 mm respectively. The total axial load is 60 kN and co-efficient of friction = 0.05. The shaft is rotating at 380 r.p.m. Intensity of pressure is not to exceed 350 kN/m². Calculate :*

(i) power lost in overcoming the friction and

(ii) number of collars required for the thrust bearing.

Sol. Given :

External radius, $r_1 = 210$ mm $= 0.21$ m

Internal radius, $r_2 = 160$ mm $= 0.16$ m

Total axial load, $W = 60$ kN $= 60 \times 10^3$ N

Co-efficient of friction, $\mu = 0.05$

Speed, $N = 380$ r.p.m.

Intensity of pressure, $p = 350$ kN/m$^2 = 350 \times 10^3$ N/m^2

Here the power lost in overcoming the friction is to be determined. Also no assumption is mentioned. Hence it is safe to assume uniform pressure.

(i) Power lost in overcoming friction

For uniform pressure, total frictional torque is given by equation (6.13) as

$$T = \frac{2}{3}\mu W\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]$$

$$= \frac{2}{3} \times 0.05 \times 60 \times 10^3 \left[\frac{0.21^3 - 0.16^3}{0.21^2 - 0.16^2}\right]$$

$$= 2000 \times \left[\frac{0.009261 - 0.004096}{0.0441 - 0.0256}\right]$$

$$= 2000 \times \frac{0.005165}{0.0185} = 558.378 \text{ Nm}$$

$\therefore$ Power lost in friction, $P = \frac{2\pi NT}{60}$

$$\therefore \quad P = \frac{2\pi \times 380 \times 558.378}{60} = 22219.8 \text{ W} = \mathbf{22.2198 \text{ kW}}. \quad \textbf{Ans.}$$

(ii) Number of collars required.

Number of collars, $n = \frac{\text{Total load}}{\text{Load per collar}}$

Now load per collar for uniform pressure is obtained from equation*(6.12), as

$$p = \frac{W^*}{\pi (r_1^2 - r_2^2)}$$

where W^* is the load per collar.

$$\therefore \quad W^* = p \times \pi (r_1^2 - r_2^2)$$

$$= 350 \times 10^3 \times \pi (0.21^2 - 0.16^2)$$

$$= 350 \times 10^3 \times \pi (0.0441 - 0.0256) = 20341.8 \text{ N}$$

$\therefore$ Number of collars $= \frac{\text{Total load}}{\text{Load per collar}}$

$$= \frac{W}{W^*} = \frac{60 \times 10^3}{20341.8} = 2.95 \simeq \mathbf{3 \text{ collars}}. \quad \textbf{Ans.}$$

6.11. FRICTION CLUTCHES

The device used to transmit the rotary motion of one shaft to another, the axes of which are coincident, is known a clutch or friction clutch. With the help of friction clutch, the power is transmitted from one shaft to another shaft which must be started and stopped frequently as in the case of automobile for automotive purposes. The engine shaft and gear box shaft is connected with the help of friction clutches.

The following types of friction clutches are mostly used :

(*i*) Disc clutch or single plate clutch, (*ii*) Multi-plate clutch, and

(*iii*) Cone clutch.

The principle of disc and cone clutches are same as that of the pivot and collar bearings. Though cone clutches and multiple-disc clutch are no longer in use for power transmission of power directly from the engine shaft by solid friction, multiple plate clutch is mostly used in automobiles. All modern cars have single plate clutch with a fabric facing on each side of the plate. The clutch plate is positioned between the flywheel and a solid plate, known as *pressure plate*.

6.11.1. Disc Clutch or Single Plate Clutch. Fig. 6.15 shows the diagram of a single plate clutch (or disc clutch) which consists of a single clutch plate with friction lining (*i.e.*, a lining of friction material) on both sides. This plate is attached to a hub (which is splined). The hub is free to move axially along the splines of the driven shaft. There is a pressure plate inside the clutch body. The pressure plate pushes the clutch plate towards the flywheel by a set of strong springs. The clutch body (or cover plate) is bolted to the flywheel. The pressure plate and the flywheel rotate with the driving shaft. The movement of the clutch pedal (not shown in Fig. 6.15) is transferred to the pressure plate through a thrust bearing.

$$= 2000 \times \frac{0.005165}{0.0185} = 558.378 \text{ Nm}$$

$\therefore$ Power lost in friction, $P = \dfrac{2\pi NT}{60}$

$$\therefore \qquad P = \frac{2\pi \times 380 \times 558.378}{60} = 22219.8 \text{ W} = \mathbf{22.2198 \text{ kW}}. \quad \textbf{Ans.}$$

(*ii*) *Number of collars required.*

Number of collars, $n = \dfrac{\text{Total load}}{\text{Load per collar}}$

Now load per collar for uniform pressure is obtained from equation*(6.12), as

$$p = \frac{W^*}{\pi (r_1^2 - r_2^2)}$$

where W^* is the load per collar.

$$\therefore \qquad W^* = p \times \pi (r_1^2 - r_2^2)$$

$$= 350 \times 10^3 \times \pi (0.21^2 - 0.16^2)$$

$$= 350 \times 10^3 \times \pi (0.0441 - 0.0256) = 20341.8 \text{ N}$$

$\therefore$ Number of collars $= \dfrac{\text{Total load}}{\text{Load per collar}}$

$$= \frac{W}{W^*} = \frac{60 \times 10^3}{20341.8} = 2.95 = \mathbf{3 \text{ collars}}. \quad \textbf{Ans.}$$

6.11. FRICTION CLUTCHES

The device used to transmit the rotary motion of one shaft to another, the axes of which are coincident, is known a clutch or friction clutch. With the help of friction clutch, the power is transmitted from one shaft to another shaft which must be started and stopped frequently as in the case of automobile for automotive purposes. The engine shaft and gear box shaft is connected with the help of friction clutches.

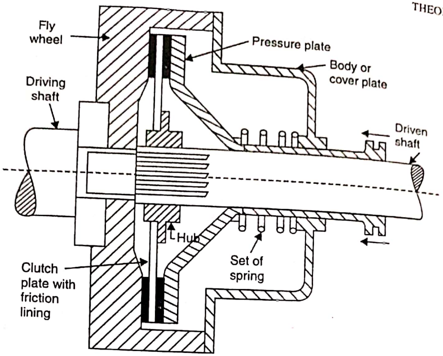
The following types of friction clutches are mostly used :

(*i*) Disc clutch or single plate clutch, (*ii*) Multi-plate clutch, and

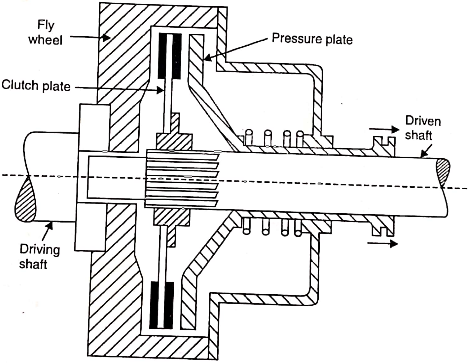
(*iii*) Cone clutch.

The principle of disc and cone clutches are same as that of the pivot and collar bearings. Though cone clutches and multiple-disc clutch are no longer in use for power transmission of power directly from the engine shaft by solid friction, multiple plate clutch is mostly used in automobiles. All modern cars have single plate clutch with a fabric facing on each side of the plate. The clutch plate is positioned between the flywheel and a solid plate, known as *pressure plate.*

6.11.1. Disc Clutch or Single Plate Clutch. Fig. 6.15 shows the diagram of a single plate clutch (or disc clutch) which consists of a single clutch plate with friction lining (*i.e.*, a lining of friction material) on both sides. This plate is attached to a hub (which is splined). The hub is free to move axially along the splines of the driven shaft. There is a pressure plate inside the clutch body. The pressure plate pushes the clutch plate towards the flywheel by a set of strong springs. The clutch body (or cover plate) is bolted to the flywheel. The pressure plate and the flywheel rotate with the driving shaft. The movement of the clutch pedal (not shown in Fig. 6.15) is transferred to the pressure plate through a thrust bearing.



(a)



(b)

Fig. 6.15

Fig. 6.15 (a) shows the position of the clutch when it is in engaged position. This position will be when the foot is taken off from the clutch pedal. The set of strong springs will move the pressure plate to bring it in contact with the clutch plate which is attached to the hub. The hub moves axially along the splines of the driven shaft. Thus the clutch plate is tightly gripped between the pressure plate and the flywheel. There is a friction lining on both sides of the clutch plate. The friction lining on one side of plate is in contact with flywheel whereas the friction lining on other side of the clutch plate is in contact with pressure plate. Due to the tightly gripping of clutch plate between pressure plate and flywheel, the clutch plate and hence driven shaft starts rotating.

Fig. 6.15 (*b*) shows the position of the clutch when it is in disengaged position. This position will be when the clutch pedal is pressed down by foot (not shown in Fig.). The set of springs will be compressed and the pressure plate will move away from flywheel. This action removes the pressure from the clutch plate. The clutch plate will move back from the flywheel. The friction linings on the clutch plate will be free of contact with the pressure plate or the flywheel. The flywheel will rotate without driving the clutch plate and thus driven shaft.

The power will be transmitted from the driving shaft to the driven shaft in engaged position. If the torque due to frictional force (provided by friction linings on the clutch plate) is more than the torque to be transmitted, there will be no slip between driving and driven shafts.

Theory of Single Plate Clutch

Refer to Fig. 6.16.

Let r_1 = External radius of friction lining on clutch plate

r_2 = Internal radius of friction lining

p = Intensity of pressure

W = Total axial load (or Axial thrust with which the friction surfaces are held together)

μ = Co-efficient of friction

T = Torque transmitted.

The theory of single plate clutch is also based on the same principle as that of collar bearing. In case of collar bearing, the power lost due to friction should be reduced and hence the value of co-efficient of friction should decrease. But in case of clutch the power transmitted by friction linings should be more and hence co-efficient of friction should be increased.

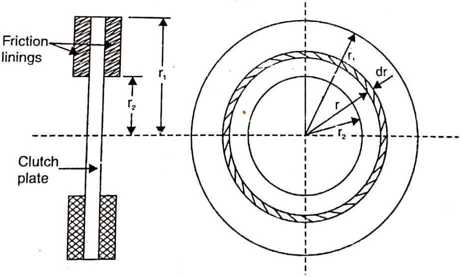


Fig. 6.16

Also in case of a new clutch, the intensity of pressure is approximately uniform over the entire surface whereas in an old clutch the uniform wear theory is more approximate.

Consider a circular ring of radius r and thickness dr as shown in Fig. 6.16.

Area of ring, $dA = 2\pi r\, dr$

Axial load on ring, dW = Pressure × Area of ring

$= p \times 2\pi r\, dr$

Frictional force on the ring, $dF = \mu \times \text{Load on ring}$

$= \mu \times (p \times 2\pi r\, dr)$

Frictional torque on ring, $dT = \text{Frictional force} \times \text{Radius}$

$= dF \times r$

$= (\mu \times p \times 2\pi r\, dr) \times r$

$= \mu \times p \times 2\pi r^2\, dr$ ...(6.15)

(i) For Uniform Pressure

$$p = \text{Constant}$$

$$\therefore \quad p = \frac{W}{\pi (r_1^2 - r_2^2)} \quad ...(6.16)$$

where W = Total axial thrust with which contact surfaces (or friction surfaces) are held together.

Total friction torque is obtained by integrating equation (6.14) from r_2 to r_1.

$\therefore$ Total friction torque acting on the friction surface,

$$T = \int_{r_2}^{r_1} dT = \int_{r_2}^{r_1} \mu \times p \times 2\pi r^2\, dr$$

$$= 2\pi\mu p \left[\frac{r^3}{3}\right]_{r_2}^{r_1} = 2\pi\mu p \left[\frac{r_1^3 - r_2^3}{3}\right]$$

$$= \frac{2}{3} \pi\mu \times \frac{W}{\pi (r_1^2 - r_2^2)} \times (r_1^3 - r_2^3) \qquad \left[\because\ P = \frac{W}{\pi (r_1^2 - r_2^2)}\right]$$

$$= \frac{2}{3} \mu W \left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right] \quad ...(6.17)$$

Total frictional torque acting on the friction surface can also be expressed in terms of mean radius (R_m) of the friction surface as

$$T = \mu W \times R_m \quad ...(6.18)$$

Comparing the above two equations, we get the value of R_m as

$$R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) \quad ...(6.19)$$

In a single clutch plate, there are two friction surfaces, one on each side of the friction plate, hence total frictional torque on the clutch plate is given by

$$T^* = 2T$$

$$= 2 \times \left[\frac{2}{3} W \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)\right] \quad ...(6.19A)$$

where T^* = Total frictional torque on the clutch plate.

(ii) For Uniform Wear

$$p \times r = \text{constant} \qquad (\text{say } C)$$

$$\therefore \quad p = \frac{C}{r}$$

We know that axial load on ring [Refer to Fig. 6.16]

$$dW = p \times 2\pi r\, dr$$

∴ Total axial load is given by integrating the above equation

$$\therefore \quad W = \int_{r_2}^{r_1} p \times 2\pi r \, dr$$

$$= \int_{r_2}^{r_1} \frac{C}{r} \times 2\pi r \, dr \qquad \left(\because \; p = \frac{C}{r}\right)$$

$$= 2\pi C \, [r]_{r_2}^{r_1} = 2\pi C \, [r_1 - r_2] \qquad \text{...(6.20)}$$

$$\therefore \quad C = \frac{W}{2\pi (r_1 - r_2)}$$

The total frictional torque on the friction surface is obtained by integrating equation (6.15) from r_2 to r_1.

$$\therefore \quad T = \int_{r_2}^{r_1} dT = \int_{r_2}^{r_1} \mu \times p \times 2\pi r^2 \, dr$$

$$= \int_{r_2}^{r_1} \mu \times \frac{C}{r} \times 2\pi r^2 \, dr = \mu \times C \times 2\pi \int_{r_2}^{r_1} r \, dr$$

$$= \mu \times C \times 2\pi \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = \mu \times C \times 2\pi \times \left[\frac{r_1^2 - r_2^2}{2}\right]$$

$$= \mu \times C \times \pi \, [r_1^2 - r_2^2] = \mu \times \frac{W}{2\pi (r_1 - r_2)} \times \pi (r_1^2 - r_2^2)$$

$$= \frac{\mu \times W}{2} \times (r_1 + r_2) \qquad \text{...(6.21)}$$

$$= \mu \times W \times R_m \qquad \text{...(6.22)}$$

where R_m = Mean radius = $\dfrac{r_1 + r_2}{2}$ ...(6.23)

∴ Total torque on a single clutch plate, is given by

$$T^* = 2T$$

$$= 2 \times \left[\frac{\mu W}{2} (r_1 + r_2)\right] \qquad \text{...(6.24)}$$

N.B. (*i*) For power transmission by friction through a clutch, uniform wear theory gives safer result. Hence uniform wear should be assumed in case of friction clutch, unless it is specified otherwise.

Problem 6.7. *Calculate the power transmitted by a single plate clutch at a speed of 2000 r.p.m., if the outer and inner radii of friction surfaces are 150 mm and 100 mm respectively. The maximum intensity of pressure at any point of contact surface should not exceed 0.8×10^5 N/m². Take both sides of the plate as effective and co-efficient of friction = 0.3. Assume uniform wear.*

Sol. Given :

Speed, N = 2000 r.p.m.

Outer radius of friction surface, r_1 = 150 mm = 0.15 m

Inner radius, r_2 = 100 mm = 0.1 m

Maximum pressure, $p_{max} = 0.8 \times 10^5$ N/m²

Co-efficient of friction, μ = 0.3

No. of effective sides = 2

For uniform wear, we have

$$p \times r = \text{constant (say} = C)$$

or

$$p_1 \times r_1 = p_2 r_2 = C$$

As for uniform wear, the product of pressure and radius is constant, hence pressure will be more where radius is less. Therefore at inner radius, the pressure will be more.

$\therefore$ $$p_{max} \times r_2 = C \quad (\because \; r_2 \text{ is inner radius})$$

or $$(0.8 \times 10^5) \times 0.1 = C \quad (\because \; p_{max} = 0.8 \times 10^5 \text{ and } r_2 = 0.1 \text{ m})$$

or $$C = 0.8 \times 10^4$$

Using equation (6.20) for uniform wear,

$$W = 2\pi C\,(r_1 - r_2)$$

$$= 2\pi \times 0.8 \times 10^4 \times (0.15 - 0.10) \text{ N} = 2513.27 \text{ N}$$

The torque due to both active surfaces is given by equation (6.24) as

$$T^* = 2 \times \left[\frac{\mu W}{2}(r_1 + r_2)\right]$$

$$= 2 \times \left[\frac{0.3 \times 2513.27}{2}(0.15 + 0.10)\right] \text{ Nm} = 188.49 \text{ Nm}$$

$\therefore$ Power transmitted by the clutch is given by

$$P = \frac{2\pi NT}{60}$$

$$= \frac{2\pi \times 2000 \times 188.49}{60} = 39477.25 \text{ W} = \mathbf{39.477 \text{ kW}}. \quad \textbf{Ans.}$$

Problem 6.8. *Determine the external and internal radii of the friction plate of a single clutch if maximum torque transmitted is 90 Nm. The external radius of the friction plate is 1.5 times the internal radius and the maximum intensity of pressure at any point of contact surface should not exceed 0.8 × 10⁵ N/m². Take both sides of the plate as effective and co-efficient of friction = 0.3. Assume uniform wear. Also calculate the axial force exerted by the springs.*

Sol. Given :

Torque, $T = 90$ Nm ; external radius = 1.5 × internal radius *i.e.*, $r_1 = 1.5\, r_2$;

$p_{max} = 0.8 \times 10^5$ N/m² ; $\mu = 0.3$,

No. of effective sides = 2.

For uniform wear, $p \times r = \text{constant}$ (say $= C$)

or $$p_1 \times r_1 = p_2 r_2 = C$$

The pressure will be maximum at the inner radius.

$\therefore$ $$p_{max} \times r_2 = C$$

or $$0.8 \times 10^5 \times r_2 = C \quad \text{...(i)}$$

Now using equation (6.20) for uniform wear,

$$W = 2\pi C\,(r_1 - r_2)$$

$$= 2\pi \times 0.8 \times 10^5 \times r_2\,(1.5\, r_2 - r_2)$$

$$[\because \text{ From } (i),\; C = 0.8 \times 10^5 \times r_2 \text{ and } r_1 = 1.5\, r_2]$$

$$= 2\pi \times 0.8 \times 10^5 \times r_2^2 \times (1.5 - 1) = 2\pi \times 0.8 \times 10^5 \times r_2^2 \times 0.5$$

$$= 251327.4\, r_2^2 \text{ N} \quad \text{...(ii)}$$

The frictional torque due to both active surfaces is given by equation (6.24) as

$$T^* = 2 \times \left[\frac{\mu W}{2}(r_1 + r_2)\right]$$

$$= 2 \times \left[\frac{0.3 \times 251327.4\, r_2^2}{2}(1.5r_2 + r_2)\right] \qquad (\because \; W = 251327.4\, r_2^2)$$

$$= 0.3 \times 251327.4\, r_2^3\,(1.5 + 1)$$

$$= 0.3 \times 251327.4 \times 2.5 \times r_2^3 \text{ Nm} \qquad ...(iii)$$

But maximum torque transmitted = 90 Nm ...(*iv*)

Hence equating the two values of the torque given by equations (*iii*) and (*iv*),

$$0.3 \times 251327.4 \times 2.5 \times r_2^3 = 90$$

$$r_2 = \left(\frac{90}{0.3 \times 251327.4 \times 2.5}\right)^{1/3}$$

$$= (4.77465 \times 10^{-4})^{1/3} = 0.07818 \text{ m} \simeq \mathbf{78.2\ mm.} \quad \textbf{Ans.}$$

$$r_1 = 1.5\, r_2 = 1.5 \times 78.2 = \mathbf{117.3\ mm.} \quad \textbf{Ans.}$$

Substituting the value of r_2 in equation (*ii*), we get

$$W = 251327.4 \times 0.07818^2 = \mathbf{1536.14\ N.} \quad \textbf{Ans.}$$

Problem 6.9 *The external radius of a friction plate of a single clutch having both sides effecti... The maximum*

6.11.2. Multi-plate Clutch. Fig. 6.17 shows the diagram of multi-plate clutch with friction plates having friction linings on both sides except the first plate which is adjacent to the flywheel. This plate is having friction lining on one side. The friction plates are connected on the top to the flywheel. Hence the friction plates rotate with the flywheel and hence with the driving shaft. The friction plates are also free to move axially.

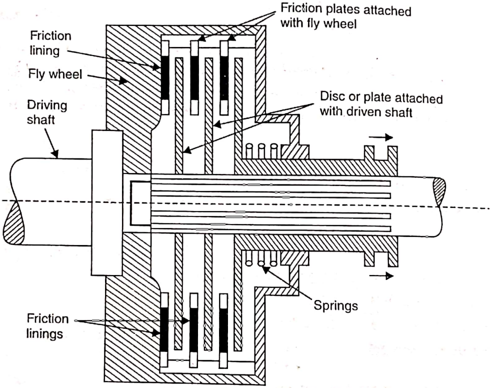


Fig. 6.17

The discs or plates are also supported on splines of the driven shaft. Hence these plates rotate with driven shaft. These plates are situated in between the friction plates and can also slide axially as shown in Fig. 6.17. Thus Fig. 6.17 shows the position of the friction plates and disc plates in disengaged position.

In the engaged position (which will be when the foot is taken off from the clutch pedal), the set of strong springs will press the discs into contact with the friction plates (or friction linings on the friction plates). Hence the power will be transmitted from the driving to the driven shaft.

Multi-plate clutch is used when a large torque is to be transmitted such as in case of motor cars and machine tools.

Theory of Multi-plate Clutch

Let r_1 = External radius of friction lining on friction plate,

r_2 = Internal radius of friction lining on friction plate,

W = Total axial load,

p = Intensity of pressure,

n_1 = Number of friction plates on driving shaft,

n_2 = Number of discs on the driven shaft

Then the number of active surfaces or friction surfaces (n) will be given as

$$n = n_1 + n_2 - 1 \quad ...(6.25)$$

Total torque transmitted is given by,

$$T = n \times \mu \times W \times R_m \quad ...(6.26)$$

where R_m = Mean radius of friction surfaces

$$= \left(\frac{r_1 + r_2}{2}\right) \quad \text{(For uniform wear)}$$

$$= \frac{2}{3}\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right] \quad \text{(For uniform pressure)}$$

Problem 6.14. *A multi-clutch has six plates (friction rings) on the driving shaft and six plates on the driving shaft. The external radius of the friction surface is 115 mm whereas the internal radius is 80 mm. Assuming uniform wear and co-efficient of friction as 0.1, find the power transmitted at 2000 r.p.m. Axial intensity of pressure is not to exceed 0.16 N/mm².*

Sol. Given :

No. of friction plates, $n_1 = 6$

No. of discs on driven shaft, $n_2 = 6$

∴ No. of active surfaces (or friction surfaces) are given as,

$$n = n_1 + n_2 - 1 = 6 + 6 - 1 = 11$$

External radius of friction surface, r_1 = 115 mm = 0.115 m

Internal radius, r_2 = 80 mm = 0.8 m

Co-efficient of friction, $\mu = 0.1$

Speed, N = 2000 r.p.m.

Max. intensity of pressure, p_{max} = 0.16 N/mm² = 0.16×10^6 N/m²

Theory assumed = Uniform wear.

Total torque transmitted is given by equation (6.26) as

$$T = n \times \mu \times W \times R_m \quad ...(i)$$

where R_m = Mean radius of friction surface

$$= \frac{r_1 + r_2}{2} \quad \text{(For uniform wear)}$$

$$= \frac{0.115 + 0.08}{2} = 0.0975 \text{ m}$$

Let us now find the value of W.

For uniform wear, $p \times r$ = constant = C

For maximum pressure, radius is minimum. Hence pressure will be maximum at internal radius.

∴ $p_{max} \times r_2 = C$

or $0.16 \times 10^6 \times 0.08 = C$

or $C = 128 \times 10^2$

The expression for axial load (W) for uniform wear is given by equation (6.20) as

$$W = 2\pi C\,(r_1 - r_2)$$

$$= 2\pi \times 128 \times 10^2\,(0.115 - 0.08) = 2814.867 \text{ N}$$

Substituting the values of W, μ, n and R_m in equation (i),

$$T = 11 \times 0.1 \times 2814.867 \times 0.0975 = 301.894 \text{ Nm}$$

∴ Power transmitted is given by,

$$P = \frac{2\pi NT}{60}$$

$$= \frac{2\pi \times 2000 \times 301.894}{60} = 63228.5 \text{ W} = \mathbf{63.2285 \text{ kW}}. \quad \textbf{Ans.}$$

Problem 6.15. *A multi-plate clutch transmits 25 kW of power at 1600 r.p.m. It has three discs on the driving shaft and two on the driven shaft. Co-efficient of friction for the friction surfaces is 0.25. The external and internal radii of friction surfaces are 100 mm and 50 mm respectively. Find the maximum intensity of pressure between the discs. Assume uniform wear.*

Sol. Given :

$P = 25$ kW $= 25 \times 10^3$ W ; $N = 1600$ r.p.m.

$\mu = 0.25$; $r_1 = 100$ mm $= 0.1$ m ; $r_2 = 50$ mm $= 0.05$ m

No. of discs on driving shaft, $n_1 = 3$; No. of discs on driven shaft, $n_2 = 2$.

∴ No. of friction (or active) surfaces, $n = n_1 + n_2 - 1 = 3 + 2 - 1 = 4$

Theory assumed = Uniform wear.

Let p_{max} = Max. intensity of pressure.

We know that, $$P = \frac{2\pi NT}{60}$$

or $$25 \times 10^3 = \frac{2\pi \times 1600 \times T}{60}$$

or $$T = \frac{25 \times 10^3 \times 60}{2\pi \times 1600} = 149.207 \text{ Nm}$$

Total torque transmitted is also given by equation (6.26) as

$$T = n \times \mu \times W \times R_m \quad \text{...}(i)$$

where R_m = Mean radius of friction surface

$$= \frac{r_1 + r_2}{2} \quad \text{(For uniform wear)}$$

$$= \frac{0.1 + 0.05}{2} = 0.075 \text{ m}$$

$n = 4$ and $\mu = 0.25$

Substituting the values of T, n, μ and R_m in equation (i), we get

$$149.207 = 4 \times 0.25 \times W \times 0.075$$

∴ $$W = \frac{149.207}{4 \times .25 \times 0.075} = 1989.426 \text{ N}$$

The expression for axial load (W) for uniform wear is also given by equation (6.20) as

$$W = 2\pi C\,(r_1 - r_2)$$

Substituting the values of W, r_1 and r_2, we get

$$1989.426 = 2\pi \times C \times (0.1 - 0.05)$$

or $$C = \frac{1989.426}{2\pi \times 0.05} = 6332.54$$

For uniform wear, $p \times r = C$ (constant)

The pressure is maximum at internal radius

$\therefore$ $$p_{max} \times r_2 = C$$

or $$p_{max} \times 0.05 = 6332.54$$

[$\because$ $C = 6332.54$ and $r_2 = 0.05$]

or $$p_{max} = \frac{6332.54}{0.05} = 1266.50 \text{ N/m}^2 = \mathbf{0.12665 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Problem 6.16. *A power of 60 kW is transmitted by a multi-plate clutch at 1500 r.p.m. Axial intensity of pressure is not to exceed 0.15 N/mm². The co-efficient of friction for the friction surfaces is 0.15. The external radius of friction surface is 120 mm. Also the external radius is equal to 1.25 times the internal radius. Find the number of plates needed to transmit the required power. Assume uniform wear.*

Sol. Given :

$P = 60$ kW $= 60 \times 10^3$ W ; $N = 1500$ r.p.m. ; $p_{max} = 0.15$ N/mm² $= 0.15 \times 10^6$ N/m² ;

$\mu = 0.15$; $r_1 = 120$ mm $= 0.12$ m ; $r_1 = 1.25 \times r_2$

$\therefore$ $$r_2 = \frac{r_1}{1.25} = \frac{0.12}{1.25} = 0.096 \text{ m.}$$

Assume uniform wear. Find the number of plates required.

For uniform wear,

$$p \times r = \text{constant (say} = C)$$

$\therefore$ Pressure will be maximum, at the internal radius

$\therefore$ $$p_{max} \times r_2 = C$$

or $$(0.15 \times 10^6) \times 0.096 = C$$

$\therefore$ $$C = 0.15 \times 10^6 \times 0.096 = 14400$$

For uniform wear, the axial thrust or load (W) is given by equation (6.20) as

$$W = 2\pi C (r_1 - r_2)$$
$$= 2\pi \times 14400 (0.12 - 0.096)$$
$$= 2171.47 \text{ N} \quad ...(i)$$

Now let us find the total torque transmitted from the given power,

$$P = \frac{2\pi NT}{60}$$

or $$60 \times 10^3 = \frac{2\pi \times 1500 \times T}{60}$$

$\therefore$ $$T = \frac{60 \times 10^3 \times 60}{2\pi \times 1500} = 381.972 \text{ Nm} \quad ...(ii)$$

But the total torque transmitted is also given by equation (6.26) as

$$T = n \times \mu \times W \times R_m \quad ...(iii)$$

where n = no. of friction surfaces or active surfaces

R_m = Mean radius of friction surfaces

$$= \frac{r_1 + r_2}{2} \quad \text{(For uniform wear)}$$

$$= \frac{0.12 + 0.096}{2} = 0.108 \text{ m}$$

and $W = 2171.47$ [From equation (*i*)]

Substituting the known values in equation (*iii*), we get

$$T = n \times 0.15 \times 2171.47 \times 0.108$$
$$= n \times 35.1778 \quad ...(iv)$$

Equating the two values of T given by equations (*ii*) and (*iv*),

$$381.972 = n \times 35.1778$$

$$\therefore \quad n = \frac{381.972}{35.1778} = 10.85 \text{ or } 11 \text{ surfaces}$$

∴ Number of friction surfaces required = **11 surfaces. Ans.**

But no. of friction surfaces,

$$n = n_1 + n_2 - 1 \quad \text{or} \quad 11 = n_1 + n_2 - 1$$

$$\therefore \quad n_1 + n_2 = 11 + 1 = 12$$

Hence there will be total 12 plates. The six plates (6) will be revolving with the driving shaft and other six with the driven shaft.

6.11.3. Cone Clutch. Fig. 6.18 shows the diagram of a cone clutch, in which the contact surfaces are in the form of cones. The driver cone is keyed to the driving shaft whereas the driven cone is keyed to the driven shaft. In the engaged position, the friction surfaces of the two cones are in complete contact due to spring pressure. In this position torque is transmitted from driving shaft to the driven shaft. For disengaging the clutch, the driven cone is pulled back through a lever system against the force of spring.

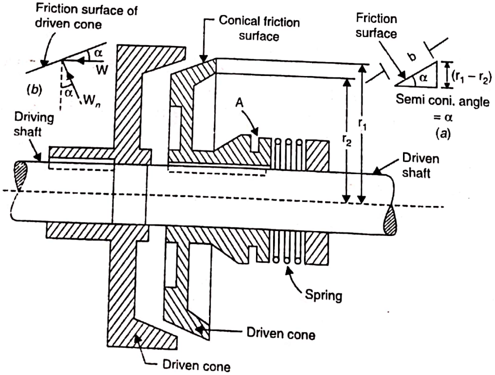


Fig. 6.18

The contact surfaces of the clutch may be metal to metal contact, but more often the driven cone surface is lined with some friction material. In action the cone clutch is similar to the truncated conical pivot.

Let r_1 = External radius of friction surface

r_2 = Internal radius of friction surface

α = Semi cone angle or the angle of the friction surface with the axis of the shaft

W = Total axial load required to engage the clutch supplied by spring

R_m = Mean radius of friction surface

μ = Co-efficient of friction

b = Width of contact surface or width of cone face

$$= \frac{(r_1 - r_2)}{\sin\alpha} \quad \text{[See Fig. 6.18 (a)]} \qquad ...(6.27)$$

(i) Case of Uniform Pressure

Similar to the truncated cone pivot, for uniform pressure we have the following equations :

$$p = \frac{W}{\pi(r_1^2 - r_2^2)}$$

or
$$W = p \times \pi (r_1^2 - r_2^2)$$

and
$$T = \frac{2}{3} \times \frac{\mu W}{\sin\alpha}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) \qquad ...(6.28)$$

(ii) Case of Uniform Wear

$$p \times r = \text{constant (say } C)$$

$$W = 2\pi C (r_1 - r_2) \qquad ...(6.29)$$

and
$$p_{max} \times r_2 = C$$

Also
$$T = \frac{1}{2}\frac{\mu W}{\sin\alpha}(r_1 + r_2) \qquad ...(6.30)$$

(iii) Driving Torque based on Mean radius

Let p_m = Intensity of pressure at mean radius normal to friction surface

W_n = Total load normal to friction surface

= (pressure normal to friction surface) × Area of friction surface based on mean radius

$$= p_n \times (2\pi R_m \times b) \qquad ...(6.31)$$

W = Component of W_n in axial direction

$$= W_n \times \sin\alpha \quad \text{[See Fig. 6.18 (b)]} \qquad ...(6.32)$$

$$T = \frac{1}{2}\frac{\mu W}{\sin\alpha}(r_1 + r_2)$$

$$= \mu \times \frac{W}{\sin\alpha} \times \left(\frac{r_1 + r_2}{2}\right)$$

$$= \mu \times W_n \times R_m \qquad ...(6.33)$$

$$\left(\because \frac{W}{\sin\alpha} = W_n ; \frac{r_1 + r_2}{2} = R_m\right)$$

Equation (6.33) gives the torque in terms of mean radius and load normal to friction surface.

Problem 6.17. *A cone clutch of cone angle 30° is used to transmit a power of 10 kW at 800 r.p.m. The intensity of pressure between the contact surfaces is not to exceed 85 kN/m². The* *width of the conical friction surface is half of the mean radius. If co-efficient of friction = 0.15, then find the dimensions of the contact surfaces. Assume uniform wear. Also find the axial load or force required to hold the clutch while transmitting the power. What is the width of the friction surface ?*

Sol. Given :

Cone angle = 30° ∴ Semi-cone angle, $\alpha = 15°$

Power, $P = 10$ kW $= 10 \times 10^3$ W ; $N = 800$ r.p.m.

Max. pressure, $p_{max} = 85$ kN/m^2 $= 85 \times 10^3$ N/m^2

Width, $b = \frac{1}{2} \times$ Mean radius $= \frac{1}{2} \times R_m = \frac{1}{2}\left(\frac{r_1 + r_2}{2}\right)$ $\left(\because \text{ For uniform wear}, R_m = \frac{r_1 + r_2}{2}\right)$

Co-efficient of friction, $\mu = 0.15$

Find : (*i*) dimensions of contact surfaces *i.e.*, r_1 and r_2.

(*ii*) Axial force of load required to keep the clutch engaging.

Assumed uniform wear.

We know that, $$P = \frac{2\pi NT}{60}$$

or $$10 \times 10^3 = \frac{2\pi \times 800 \times T}{60}$$

∴ $$T = \frac{60 \times 10 \times 10^3}{2\pi \times 800} = 119.366 \text{ Nm} \quad ...(i)$$

Now width '*b*' is given as

$$b = \frac{1}{2}\left(\frac{r_1 + r_2}{2}\right) = \frac{r_1 + r_2}{4} \quad ...(ii)$$

Also the value of '*b*' from equation (6.27) is given as

$$b = \frac{r_1 - r_2}{\sin \alpha} = \frac{r_1 - r_2}{\sin 15°} = \frac{r_1 - r_2}{0.2588} \quad ...(iii)$$

Hence equating the two values of '*b*' given by equations (*ii*) and (*iii*),

$$\frac{r_1 + r_2}{4} = \frac{r_1 - r_2}{0.2588}$$

or $$r_1 + r_2 = \frac{4}{0.2588}(r_1 - r_2)$$

$$= 15.456\,(r_1 - r_2) = 15.456\,r_1 - 15.456\,r_2$$

or $$16.456\,r_2 = 14.456\,r_1$$

or $$r_1 = \frac{16.456}{14.456}\,r_2$$

or $$r_1 = 1.138 r_2 \quad ...(iv)$$

Now let us find the value of W (axial load) for uniform wear.

For uniform wear, $p \times r = C$ (constant)

The pressure will be maximum at internal radius

∴ $$p_{max} \times r_2 = C \quad ...(v)$$

or $$85 \times 10^3 \times r_2 = C$$

The value of W for uniform wear is given by equation (6.29) as

$$W = 2\pi C (r_1 - r_2)$$
$$= 2\pi \times 85 \times 10^3\, r_2 (r_1 - r_2) \quad [\because \text{ From } (v),\ C = 85 \times 10^3 \times r_2]$$
$$= 534070\, r_2 (r_1 - r_2) \quad ...(vi)$$

The frictional torque for uniform wear is given by equation (6.30), as

$$T = \frac{1}{2} \frac{\mu W}{\sin\alpha} (r_1 + r_2)$$
$$= \frac{1}{2} \times \frac{0.15 \times 534070 r_2 (r_1 - r_2)}{\sin 15°} (r_1 + r_2)$$
$$[\because \text{ From } (vi),\ W = 534070\, r_2 (r_1 - r_2)]$$
$$= 154762 r_2 (r_1 - r_2)(r_1 + r_2) \quad ...(vii)$$

Substituting the value of T from equation (i) and value of r_1 from equation (iv) in the above equation, we get

$$119.366 = 154762 r_2 (r_1^2 - r_2^2)$$
$$= 154762 r_2 [(1.138 r_2)^2 - r_2^2] \quad (\because r_1 = 1.138 r_2)$$
$$= 154762 r_2 (1.295044 r^2 - r_2^2) = 45661 r_2^3$$

$$\therefore \quad r_2 = \left(\frac{119.366}{45661}\right)^{1/3} = 0.138 \text{ m} = \mathbf{138\ mm.} \quad \textbf{Ans.}$$

$$\therefore \quad r_1 = 1.138 r_2$$
$$= 1.138 \times 0.138 = 0.157 \text{ m} = \mathbf{157\ mm.} \quad \textbf{Ans.}$$

Substituting the values of r_1 and r_2 in equation (vi), we get

$$W = 534070 r_2 (r_1 - r_2)$$
$$= 534070 \times 0.138 (0.157 - 0.138) = \mathbf{1400.3\ N.} \quad \textbf{Ans.}$$

Width of the friction surface is given by equation (6.27) as

$$b = \frac{r_1 - r_2}{\sin\alpha} = \frac{157 - 138}{\sin 15°} = \frac{19}{0.2588} = \mathbf{73.4\ mm.} \quad \textbf{Ans.}$$

8

Brakes and Dynamometers

8.1. INTRODUCTION

A *brake* is a device used either to bring to rest a body which is in motion or to hold a body in a state of rest or of uniform motion against the action of external forces or couples. Actually the brake offers the frictional resistance to the moving body and this frictional resistance retards the motion and the body comes to rest. In this process, the kinetic energy of the body is absorbed by brakes.

A *dynamometer* is a device used to measure the frictional resistance or frictional torque. This frictional resistance (or frictional torque) is obtained by applying a brake. Hence dynamometer is also a brake in addition it has a device to measure the frictional resistance (or frictional torque). This chapter deals with different types of brakes and dynamometers.

8.2. TYPES OF BRAKES

The brakes are classified as :

(*a*) Hydraulic brakes,

(*b*) Electric brakes,

(*c*) Mechanical brakes.

This chapter deals with mechanical brakes only. The following are the important types of mechanical brakes :

(*i*) Simple block or Shoe brake.

(*a*) Single block or Shoe brake

(*b*) Double block or Shoe brake.

(*ii*) Band brake

(*iii*) Band and block brake.

(*iv*) Internal expanding shoe brake.

8.2.1. Simple Block or Shoe Brake. A simple arrangement for applying a braking force is shown in Fig. 8.1. The face of a brake has a special friction material which has a high value of co-efficient of friction.

A single block or shoe brake consists of a block or shoe which is pressed against a rotating drum as shown in Fig. 8.1. The block is rigidly fixed to the lever. The force is applied at one end of the lever and the other end of the lever is pivoted on a fixed fulcrum *O*. As the force is applied to the lever, the block is pressed against the rotating drum. The friction between the block and the drum causes a tangential force to act on the drum, which tends to prevent its rotation.

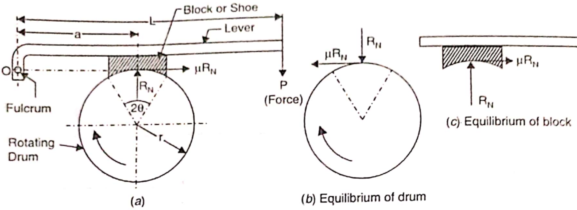


Fig. 8.1

The block is made of a softer material than that of the drum so that the block can be replaced easily on wearing. For light and slow vehicles, wood and rubber are used whereas for heavy and flat vehicles, cast steel is used.

Let P = Force applied at the lower end

r = Radius of the drum

μ = Co-efficient of friction

R_N = Normal reaction on the block

2θ = Angle made by contact surface of the block at the centre of the drum

F^* = Frictional force acting on block = μR_N

T_B = Braking torque.

When force P is applied at the lever end, the block is pressed against the rotating drum. The block exerts a radial force on the drum (*i.e.* this force passes through the centre of the drum). The drum will exert a normal reaction (R_N) on the block. Hence the radial force on the drum will be equal to the normal reaction (R_N) on the block.

Assuming that the normal reaction R_N and the frictional force F^* ($= \mu R_N$) act at the mid-point of the block, we have

Braking torque on the drum = Frictional force × radius

or $T_B = F^* \times r$

$= \mu R_N \times r$ $\quad (\because \; F^* = \mu \times R_N)$ ...(8.1)

The braking torque can be calculated if the value of R_N is known in equation (8.1). The value of R_N is obtained by considering the *equilibrium of the block*.

In Fig. 8.1, the drum is rotating clock-wise. Hence the frictional force on the drum will be acting in the opposite direction [*i.e.* in the anti-clockwise direction as shown in Fig. 8.1 (*b*)]. The frictional force on the block will be opposite to the direction of the frictional force on the drum. Hence the frictional force on the block will be in the clock-wise direction as shown in Fig. 8.1 (*c*) (*i.e.*, in the same direction in which drum is rotating). Let the line of action of this frictional force (μR_N) passes through the fulcrum O of the lever. The forces acting on the block are :

(*i*) R_N (Normal reaction), (*ii*) μR_N (Frictional force), (*iii*) P (Applied force). Taking moments of all forces about the pivot O, we have

$R_N \times a = P \times L$ (The frictional force μR_N passes through O, hence its moment is zero)

$$\therefore \qquad R_N = \frac{P \times L}{a}$$

Substituting this value of R_N is equation (8.1), we get the braking torque as,

$$T_B = \mu \times \frac{P \times L}{a} \times r \qquad ...(8.2)$$

Equation (8.2) gives the value of braking torque when the line of action of the frictional force passes through the fulcrum O of the lever.

It is not necessary that the line of action of the frictional force ($\mu \times R_N$) should pass through the fulcrum O of the lever. The line of action of the frictional force may be at a distance b below or above the fulcrum O.

Let us consider these two cases :

Case 1. *When the line of action of the frictional force (μR_N) is at a distance 'b' below the fulcrum O and the drum rotates clockwise as shown in Fig. 8.2.*

The forces acting on the block are : (i) R_N acting upwards, (ii) μR_N frictional force on block acting in the same direction in which drum is rotating and (iii) P (acting downwards.)

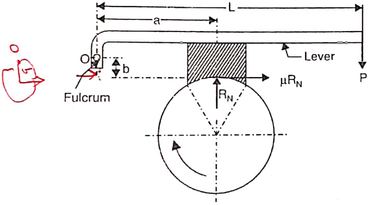


Fig. 8.2

Taking moments about the fulcrum O, we get

$$R_N \times a + \mu R_N \times b = P \times L$$

or

$$R_N (a + \mu \times b) = P \times L$$

$$\therefore \qquad R_N = \frac{P \times L}{(a + \mu b)}$$

Substituting this value of R_N is equation (8.1), we get braking torque (T_B) as,

$$T_B = \mu \times \frac{P \times L}{(a + \mu b)} \times r$$

$$= \frac{\mu \times P \times L \times r}{(a + \mu b)} \qquad ...(8.3)$$

Now consider the above case when drum is rotating in anti-clockwise direction. If the drum is rotating in anti-clockwise direction as shown in Fig. 8.3 then the frictional force $\mu \times R_N$ will also be acting in anti-clockwise direction. The moment of all forces acting on the block (*i.e.* R_N, μR_N and P) about the fulcrum O will give,

$$R_N \times a = P \times L + \mu R_N \times b$$

or

$$R_N \times a - \mu R_N \times b = P \times L$$

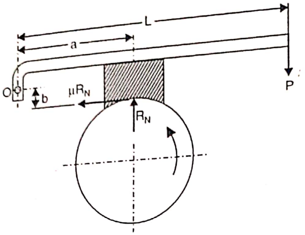


Fig. 8.3

or

$$R_N(a - \mu b) = P \times L$$

$$R_N = \frac{P \times L}{(a - \mu b)} \quad \text{...(8.4)}$$

Substituting the above value in equation (8.1), we get braking torque T_B, as

$$T_B = \mu \times \frac{P \times L}{(a - \mu b)} \times r$$

$$= \frac{\mu \times P \times L \times r}{(a - \mu b)} \quad \text{...(8.5)}$$

Consider the equation (8.4) again. From equation (8.4), the expression for the force (P) required to apply the brake is obtained as

$$P = \frac{R_N(a - \mu b)}{L} \quad \text{...(8.5A)}$$

In equation (8.5A) if, $a \le \mu b$, then P will be negative or zero. This means that no-external force is required to apply the brake and hence the brake is **self-locking.** Hence the condition for the brake to be self-locking is

$$a \le \mu b \quad \text{...(8.5B)}$$

Again consider equation (8.4). From equation (8.4), we have the value of P as

$$P = \frac{R_N(a - \mu b)}{L}$$

$$= \frac{R_N \times a - \mu R_N \times b}{L} \quad \text{...(8.5C)}$$

In the above equation '$R_N \times a$' is the moment of R_N about the fulcrum O whereas '$\mu R_N \times b$' is the moment of frictional force about the fulcrum. This moment is having negative sign.

Hence in this case the force P required to apply the brake decreases due to frictional force. Or in other words, the frictional force helps to apply the brake. Such types of brakes are known as **self-energised brakes.** In actual practice the brake should be self-energising and not self-locking. For the above case, the self-locking brake and self-energised brakes are possible. If $P = 0$ it is a self-locking brake. If $P > 0$ it is self-energising brake.

Case 2. *When the line of action of the frictional force* (μR_N) *is at a distance 'b' above the fulcrum O and the drum rotates clockwise as shown in Fig. 8.4.* The forces acting are : (*i*) R_N, (*ii*) μR_N and (*iii*) P. The frictional force ($\mu \times R_N$) on block is acting in the direction of rotation of drum. Taking the moments about the fulcrum, we get

$$R_N \times a = P \times L + \mu R_N \times b$$

$$R_N \times a - \mu R_N \times b = P \times L$$

or $$R_N (a - \mu \times b) = P \times L$$

or

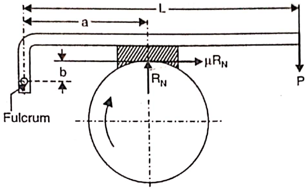


Fig. 8.4

∴ $$R_N = \frac{P \times L}{(a - \mu b)}$$

Substituting this value of R_N in equation (8.1), we get braking torque (T_B) as

$$T_B = \mu \times \frac{P \times L}{(a - \mu b)} \times r$$

$$= \frac{\mu \times P \times L \times r}{(a - \mu b)} \quad \text{...(8.6)}$$

In this case also, the brake may be self-locking or self-energised. If $P = 0$, the brake is self-locking and if $P > 0$ the brake is self-energised.

If the drum is rotating in anti-clockwise direction as shown in Fig. 8.5, then frictional force (μR_N) will also be acting in anti-clockwise direction.

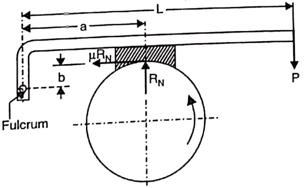


Fig. 8.5

Taking the moments of all forces about the fulcrum, we get

$$R_N \times a + \mu R_N \times b = P \times L$$

or $$R_N (a + \mu b) = P \times L$$

or $$R_N = \frac{P \times L}{(a + \mu b)}$$

Substituting this value in equation (8.1), we get braking torque as

$$T_B = \mu \times \frac{P \times L}{(a + \mu b)} \times r$$

$$= \frac{\mu \times P \times L \times r}{(a + \mu b)} \quad ...(8.7)$$

For all the above expressions, the normal reaction (R_N) and force of friction (μR_N) are assumed to be acting at the mid-point of the block. This is true only if the angle made by contact surface of the block at the centre of the rotating drum is less than or equal to 40° *i.e.* $2\theta \le 40°$. But if angle of contact is more that 40°, the normal pressure is less at the ends than at the centre. In that case, μ has to be replaced by an equivalent co-efficient of friction μ′ as given by

$$\mu' = \frac{4\mu \sin \theta}{2\theta + \sin 2\theta} \quad \text{or} \quad \mu \left[\frac{4 \sin \theta}{2\theta + \sin 2\theta} \right] \quad ...(8.8)$$

where μ = Actual co-efficient of friction.

Note. (*i*) For Case 1, self-locking of brake takes place if brake drum is rotating anti-clockwise direction.

(*ii*) For Case 2, brake will be self locking if brake drum rotates clockwise.

Problem 8.1. *The brake drum of a single block brake is rotating at 500 r.p.m. in the clockwise direction. The diameter of the drum is 400 mm and the single block brake is of the type as shown in Fig. 8.2. The force required at the end of the lever to apply the brake is 300 N. If angle of contact is 30° and L = 1 m, a = 300 mm and b = 25 mm then determine the braking torque. The co-efficient of friction is equal to 0.3.*

Sol. Given (Refer to Fig. 8.2)

Speed, $N = 500$ r.p.m.

Dia. of drum $= 400 \text{ mm} = 0.4 \text{ m}$

∴ Radius of drum, $r = \frac{400}{2} = 200 \text{ mm} = 0.2 \text{ m}$

Force at the end of lever, $P = 300$ N

Angle of contact, $2\theta = 30°$

Length of lever from fulcrum, $L = 1$ m

Distance of centre of the block from fulcum,

$a = 300 \text{ mm} = 0.3 \text{ m}$

Perpendicular distance between line of action of frictional force and fulcrum,

$b = 25 \text{ mm} = 0.025 \text{ m}$

Rotation of drum = clockwise.

Co-efficient of friction, $\mu = 0.3$

Taking the moments of all forces (R_N, μR_N and P) about fulcrum, we get

$$R_N \times a + \mu R_N \times b = P \times L$$

or $$R_N \times 0.3 + 0.3 \times R_N \times 0.025 = 300 \times 1$$

or $$R_N (0.3 + 0.3 \times 0.025) = 300$$

or $$R_N = \frac{300}{0.3 + 0.3 \times 0.025}$$

$$= \frac{300}{0.3075} = 975.6 \text{ N}$$

Braking torque (T_B) is given by equation (8.1) as

$$T_B = \mu R_N \times r$$

$$= 0.3 \times 975.6 \times 0.2 = \mathbf{58.536 \text{ Nm.}}$$ **Ans.**

Alternately

When the drum is rotating clock-wise and line of action of the frictional force is at a distance 'b' below the fulcrum (Refer to Fig. 8.2), the braking torque is given by equation (8.3). Hence using equation (8.3), we get

$$T_B = \frac{\mu \times P \times L \times r}{(a + \mu b)}$$

$$= \frac{0.3 \times 300 \times 1 \times 0.2}{(0.3 + 0.3 \times 0.025)}$$

$$= \frac{18}{0.3075} = \mathbf{58.536\ Nm.} \quad \textbf{Ans.}$$

Problem 8.2. *If the brake drum in problem 8.1 rotates in the anti-clockwise direction as shown in Fig. 8.3 and all other data remain the same, then determine : (i) the braking torque and (ii) value of 'b' for self-locking of the brake.*

Sol. Refer to Fig. 8.3.

The data from Problem 8.1 :

N = 500 r.p.m., r = 0.2 m, P = 300 N, L = 1 m, a = 0.3 m, b = 0.025 m and μ = 0.3.

(i) Braking Torque (T_B)

When the brake drum is rotating anti-clockwise and line of action of frictional force (μR_N) is at a distance 'b' below the fulcrum as shown in Fig. 8.3, the braking torque is given by equation (8.5). Hence using equation (8.5), we get

$$T_B = \frac{\mu \times P \times L \times r}{(a - \mu b)}$$

$$= \frac{0.3 \times 300 \times 1 \times 0.2}{0.3 - 0.3 \times 0.025}$$

$$= \frac{18}{0.2925} = \mathbf{61.538\ Nm.} \quad \textbf{Ans.}$$

(ii) Value of 'b' for self-locking of the brake

The condition for self-locking of the brake, is given by equation (8.5B) as

$$a \le \mu b$$

The values of 'a' and 'μ' are given. For self-locking of the brake, the value of 'b' is to be obtained. Substituting the values of a and μ in the equation, we get

$$0.3 \le 0.3 \times b$$

or $$\frac{0.3}{0.3} \le b$$

or $$1 \le b$$

or $$b \ge 1 \text{ m.} \quad \textbf{Ans.}$$

Problem 8.3. *The brake drum of a single block brake of diameter 300 mm is rotating at 400 r.p.m. as shown in Fig. 8.6. The force required at the end of the lever to apply the brake is 600 N. If angle of contact is 90° and co-efficient of friction between the drum and brake block is 0.3, find the braking torque.*

Sol. Given :

P = 600 N ; d = 300 mm or r = 150 mm = 0.15 m ; 2θ = 90° ;

μ = 0.3 ; b = 40 mm = 0.04 m ; a = 250 mm = 0.25 m ; L = 550 mm = 0.55 m.

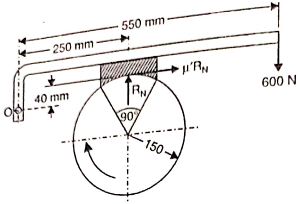


Fig. 8.6

As the angle of contact is more than 40°, hence equivalent co-efficient of friction (μ') is given by equation (8.8) as

$$\mu' = \mu\left[\frac{4\sin\theta}{2\theta + \sin 2\theta}\right]$$

$$= 0.3\left[\frac{4 \times \sin 45°}{\frac{\pi}{2} + \sin 90°}\right] \qquad \left(\because\ 2\theta = 90° = \frac{\pi}{2} \text{ radians and } \theta = 45°\right)$$

$$= \frac{0.3 \times 4 \times 0.7071}{1.5708 + 1} = 0.33$$

The forces acting on the block are :

(*i*) Normal reaction, R_N

(*ii*) Frictional force $= \mu' \times R_N = 0.33 \times R_N$

(*iii*) Applied force, $P = 600$ N

Taking moments of all forces about the fulcrum O, we get

$$R_N \times 250 = \mu' \times R_N \times 40 + 600 \times 550$$

$$250R_N = 0.33 \times 40 \times R_N + 330000$$

$$= 13.20R_N + 330000$$

or $$250R_N - 13.2R_N = 330000$$

or $$236.8R_N = 330000$$

or $$R_N = \frac{330000}{236.8} = 1393.58 \text{ N}$$

Braking torque (T_B) is given by equation (8.1) as

$$T_B = \text{Frictional force} \times \text{radius of drum}$$

$$= (\mu' \times R_N) \times (r)$$

$$= (0.33 \times 1393.58) \times (0.15) = \mathbf{68.98\ Nm.}$$ **Ans.**

Problem 8.4. *The wheels of a bicycle are of diameter 800 mm. A rider on this bicycle is travelling at a speed of 16 km/hr on a level road. The total mass of rider and bicycle is 110 kg. A brake is applied to the rear wheel. The pressure applied. On the brake is 100 N and co-efficient of friction is 0.06. Before the cycle comes to rest, find :*

(*i*) *distance travelled by the bicycle and*

(*ii*) *number of turns of its wheel.*

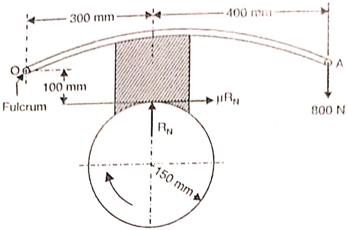


Fig. 8.7

$\therefore$ $$R_N = \frac{800 \times 700}{750} = 746.67 \text{ N}$$

Braking torque = Frictional force × radius

$= (\mu R_N) \times r$

$= 0.35 \times 746.67 \times 0.15$ Nm = **39.2 Nm. Ans.**

Double Block or Shoe Brake

When a single block brake is pressed against a rotating drum, a side thrust on the bearing of the shaft supporting the drum will act due to normal reaction (R_N). This produces the bending of the shaft. This can be prevented by using two blocks on the two sides of the drum as shown in Fig. 8.8. The braking torque becomes two times. The braking torque is given by

$$T_B = \mu R_{N1} \times r + \mu R_{N2} \times r$$
$$= (\mu R_{N1} + \mu R_{N2}) \times r$$

The value of R_{N1} is obtained by taking moments of the forces R_{N1}, μR_{N1} and P about fulcrum O_1. Similarly the value of R_{N2} is obtained by taking moments of the forces R_{N2}, μR_{N2} and P about fulcrum O_2.

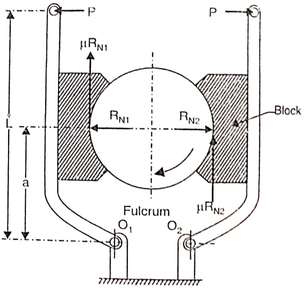


Fig. 8.8

8.2.2. Band Brake. If band is used for bringing a rotating body to rest, then it is known a band brake. A band brake may be a simple band brake or a differential brake.

(*a*) **Simple Band Brake.** It consists of one or more ropes, belt or flexible steel band lined with friction material, which embraces a part of the circumference of the rotating drum. Fig. 8.9 shows a simple band brake in which one end of the band is *attached with the fulcrum* (or fixed pin) of the lever while the other end is attached to the lever at a distance '*a*' from the fulcrum. In order to apply the brake, the band is tightened round the drum and the friction between the band and the drum provides the tangential braking torque.

The force P is applied at the free end of the lever which turns about the fulcrum O. This tightens the band on the drum and hence the brakes are applied. The braking force is provided by the friction between the band and the drum. The force P at the end of the lever for clockwise rotation and anti-clockwise rotation of drum is obtained as explained below :

Let θ = Angle of lap of the band on the drum,
T_1 = Tension in the tight side of the band,
T_2 = Tension in the slack side of the band,
r = Radius of the drum,
μ = Co-efficient of friction between band and the drum,
t = Thickness of band,

r_e = Effective radius of the drum = $\left(r + \frac{t}{2}\right)$, and

P = Force at the end of the lever.

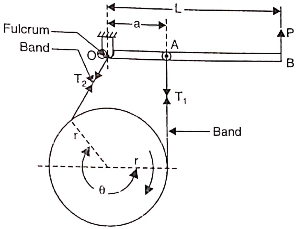


Fig. 8.9

Limiting ratio of tensions is given by,

$$\frac{T_1}{T_2} = e^{\mu\theta} \quad ...(8.9)$$

or

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu\theta \quad ...(8.9A)$$

Net torque on drum $= T_1 \times r - T_2 \times r$

$= (T_1 - T_2) \times r$

This is also the braking torque on the drum

$\therefore$ Braking torque on the drum is given by

$T_B = (T_1 - T_2) \times r$ ...if thickness of belt is neglected

$= (T_1 - T_2) \times r_e$ ...if thickness of belt is considered

where r_e = Effective radius of band.

(*i*) **Value of P for Clock-wise rotation of drum.** For clock-wise rotation drum as shown in Fig. 8.9, the end of the band connected to the fulcrum O will be slack side with tension T_2 and the end of the band attached to A will be tight side with tension T_1.

Taking moments about the fulcrum O, we get

$$P \times L = T_1 \times a \quad (\because \ T_2 \text{ passes through } O) \quad ...(8.10)$$

where L = Distance OB and a = perpendicular distance from O to the line of action of T_1.

(*ii*) **Value of P for Anti-clockwise rotation of drum.** For anti-clockwise rotation of the drum as shown in Fig. 8.10, the end of the band connected to the fulcrum O will be tight side with tension T_1 and the end of the band attached to A will be slack side with tension T_2. Taking the moments about the fulcrum O, we get

$$P \times L = T_2 \times a \qquad (\because \; T_1 \text{ passes through } O) \qquad ...(8.11)$$

where L = Length of lever from fulcrum *i.e.* distance OB

a = Perpendicular distance from O to the line of action of T_2.

Note. For simple band brake, one end of band is always connected to the fulcrum.

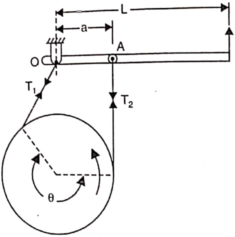


Fig. 8.10

Problem 8.6. *A simple band brake is applied to a rotating drum of diameter 500 mm. The angle of lap of the band on the drum is 270°. One end of the band is attached to a fulcrum pin of the lever and other end is to a pin 100 mm from the fulcrum. If the co-efficient of friction is 0.25 and a braking force of 90 N is applied at a distance of 600 mm from the fulcrum, find the braking torque when the drum rotates in the (i) anti-clockwise direction, and (ii) clockwise direction.*

Sol. Given :

Simple band brake. This means one end of brake is connected to fulcrum. Other data is :

d = 500 mm = 0.5 m ; r = 0.25 m ; $\theta = 270° = 270 \times \dfrac{\pi}{180} = 4.713$ rad ;

Distance a = 100 mm = 0.1 m ; L = 600 mm = 0.6 m ; μ = 0.25 ; P = 90 N.

Let T_B = braking torque.

(i) Drum rotates in anti-clockwise direction

Refer to Fig. 8.10 in which the drum is rotating in anti-clockwise direction. The braking torque is the net torque on the drum and it is given by,

$$T_B = (T_1 - T_2) \times r \qquad ...(i)$$

In the above equation, the value of 'r' is known. But the values of T_1 and T_2 are unknown. Let us first find the values of T_1 and T_2.

Taking the moments of all forces (shown in Fig. 8.10) about O

$$T_2 \times a = P \times L$$

or

$$T_2 = \frac{P \times L}{a}$$

$$= \frac{90 \times 0.6}{0.1} = 540 \text{ N}$$

Now using equation (8.9) for limiting ratio of tensions,

$$\frac{T_1}{T_2} = e^{\mu \times \theta}$$

or $$2.3 \log \left(\frac{T_1}{T_2}\right) = \mu \times \theta$$

or $$\log \left(\frac{T_1}{T_2}\right) = \frac{\mu \times \theta}{2.3}$$

$$= \frac{0.25 \times 4.713}{2.3} = 0.5123$$

$\therefore$ $$\frac{T_1}{T_2} = \text{Antilog of } 0.5123 = 3.253$$

or $$T_1 = 3.253 \times T_2$$

$$= 3.253 \times 540 \qquad (\because \; T_2 = 540 \text{ N})$$

$$= 1756.62 \text{ N}$$

Substituting the values of T_1, T_2 and r in equation (*i*), we get the braking torque as

$$T_B = (T_1 - T_2) \times r$$

$$= (1756.62 - 540) \times 0.25 \text{ Nm}$$

$$= \mathbf{304.155 \text{ Nm.}} \quad \textbf{Ans.}$$

(*ii*) *Drum rotates in clockwise direction*

Refer to Fig. 8.9 in which drum is rotating in clockwise direction. The braking torque is given by,

$$T_B = (T_1 - T_2) \times r$$

Let us first find the values of T_1 and T_2. Taking moments of all forces shown in Fig. 8.9 about O, we get

$$T_1 \times a = P \times L$$

or $$T_1 = \frac{P \times L}{a}$$

$$= \frac{90 \times 0.6}{0.1} = 540 \text{ N}$$

Also the know that

$$\frac{T_1}{T_2} = e^{\mu \times \theta}$$

or $$2.3 \log \frac{T_1}{T_2} = \mu \times \theta$$

or $$\log \left(\frac{T_1}{T_2}\right) = \frac{\mu \times \theta}{2.3}$$

$$= \frac{0.25 \times 4.713}{2.3} = 0.5123$$

$\therefore$ $$\frac{T_1}{T_2} = \text{Antilog of } 0.5123 = 3.253$$

$$\therefore \quad T_2 = \frac{T_1}{3.253} = \frac{540}{3.253} = 166 \text{ N}$$

$\therefore$ Braking torque is given by,

$$T_B = (T_1 - T_2) \times r$$
$$= (540 - 166) \times 0.25 = \mathbf{93.5\ Nm.} \quad \textbf{Ans.}$$

Problem 8.7. *Fig. 8.11 shows a simple band brake which is applied to a shaft carrying a flywheel (i.e. rotating drum) of mass 300 kg and of radius of gyration 350 mm. The flywheel rotates at 200 r.p.m. The brake drum diameter is 260 mm and co-efficient of friction is 0.20. The angle of lap of the band on the drum is 210°. If the braking torque is 39 Nm, find :*

(i) the force applied at the lever end,

(ii) the number of turns of the flywheel before it comes to rest,

(iii) the time taken by the flywheel to come to rest.

Sol. Given :

For a simple band brake, one end of the band should be connected to the fulcrum whereas the other end of the band may be connected to the lever either towards the same side is which force P is acting or towards the opposite side in which P is acting. Here the other end is in opposite direction.

The other given data is :

mass, m = 300 kg ; radius of gyration, k = 350 mm = 0.35 m ;

N = 200 r.p.m. ; d = 260 mm ; r = 130 mm = 0.13 m ;

$\mu = 0.20$; $\theta = 210°$ or $210 \times \frac{\pi}{180}$ rad = 3.666 rad.

braking torque, T_B = 39 Nm

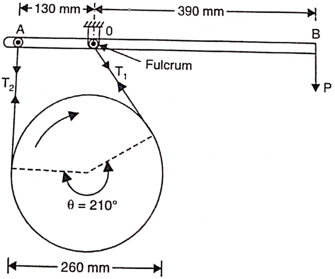


Fig. 8.11

(i) Force applied at the end of the lever

Let P = Force applied at the lever end.

The braking torque is given by,

$$T_B = (T_1 - T_2) \times r$$
$$39 = (T_1 - T_2) \times 0.13$$
$$(T_1 - T_2) = \frac{39}{0.13} = 300 \text{ N} \quad \text{...(i)}$$

Let us now find the tensions T_1 and T_2. We know that

$$\frac{T_1}{T_2} = e^{\mu \times \theta}$$

or $$2.3 \log \frac{T_1}{T_2} = \mu \times \theta = 0.2 \times 3.666 = 0.7322$$

∴ $$\log \frac{T_1}{T_2} = \frac{0.7322}{2.3} = 0.3188$$

or $$\frac{T_1}{T_2} = \text{Antilog of } 0.3188 = 2.08$$

or $$T_1 = 2.08 T_2 \quad \ldots(ii)$$

Substituting the value of T_1 in equation (*i*), we get

$$2.08T_2 - T_2 = 300$$

or $$1.08T_2 = 300$$

∴ $$T_2 = \frac{300}{1.08} = 277.77 \text{ N}$$

Substituting this value of T_2 in equation (*ii*), we get

$$T_1 = 2.08 \times 277.77 = 577.76 \text{ N}$$

To find the value of P, take the moments of all forces (*i.e.* T_1, T_2 and P) about the fulcrum O.

∴ $$P \times 390 = T_2 \times 130 \quad (\because T_1 \text{ passes through } O)$$

∴ $$P = \frac{T_2 \times 130}{390} = \frac{277.77 \times 130}{390} = \mathbf{92.59 \text{ N.}} \quad \textbf{Ans.}$$

(*ii*) *Number of turns of the flywheel before it comes to rest.*

Let n = Number of turns of the flywheel before it comes to rest.

The kinetic energy of the rotation of the flywheel is used to overcome the workdone due to braking torque (T_B), before the flywheel comes to rest.

Now K.E. of the rotation* of flywheel

$$= \frac{1}{2} \times I * \omega^2$$

$$= \frac{1}{2} \times mk^2 \times \omega^2 \quad (\because I = mk^2)$$

$$= \frac{1}{2} \times mk^2 \times \left(\frac{2\pi N}{60}\right)^2 \quad \left(\because \omega = \frac{2\pi N}{60}\right)$$

$$= \frac{1}{2} \times 300 \times 0.35^2 \times \left(\frac{2\pi \times 200}{60}\right)^2$$

$$= 8060.17 \text{ Nm} \quad \ldots(iii)$$

∴ Work done by the braking torque in 'n' number of turns of the flywheel

$$= T_B \times \text{Angular displacement in } n \text{ turns}$$

*K.E. due to linear velocity = $\frac{1}{2} mV^2$ whereas the K.E. due to rotation = $\frac{1}{2} I \times \omega^2$ where $I = mk^2$.

$= T_B \times 2\pi \times n$ ($\because$ Angular displacement for one turn $= 2\pi$) ...(iv)

$= 39 \times 2\pi \times n$

But K.E. of flywheel = Work done by braking torque

$\therefore$ $8060.17 = 39 \times 2\pi \times n$

$\therefore$ $n = \dfrac{8060.17}{39 \times 2\pi}$ = **32.89. Ans.**

(iii) Time taken by flywheel to come to rest after applying the brake

$N = 200$ r.p.m.

This means that 200 revolutions are made in one minute. The flywheel comes to rest after applying the brake in 32.89 revolution. Let us find the time for 32.89 revolution.

Time for 200 revolution = 1 minute

Time for 1 revolution $= \dfrac{1}{200}$ min

Time for 32.89 revolution $= \dfrac{1}{200} \times 32.89 = 0.16445$ min

$= 0.16445 \times 60$ seconds

= **9.867 seconds. Ans.**

Problem 8.8. *Fig. 8.12 shows a simple band brake which is applied on a drum of diameter 400 mm. The drum is rotating at 180 r.p.m. The angle of lap of the band on the drum is 270° and co-efficient of friction is 0.25. One end of the band is attached to a fixed pin (i.e. fulcrum) and other end to the lever arm at a distance of 100 mm from the fulcrum. The lever length is 600 mm. The lever arm is placed perpendicular to the diameter that bisects the angle of contact. Determine :*

(i) the necessary force required at the end of the lever arm to stop the drum if a power of 30 kW is being absorbed. Also find the direction of this force.

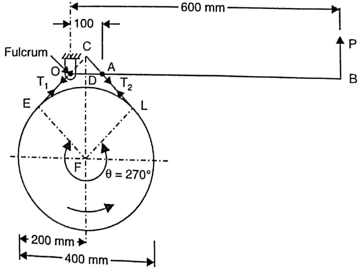


Fig. 8.12

(ii) width of steel band if maximum tensile stress in the band is not to exceed 50 N/mm². Take thickness of band as 3 mm.

Sol. Given :

Simple band brake. This means that one end of the band is connected to the fixed pin *O* (*i.e.* fulcrum). The other data is :

Let us find first the distance OC.

Distance $$OD = \frac{OA}{2} = \frac{100}{2} = 50 \text{ mm}$$

$$\angle EFC = \frac{1}{2}[360 - 270] = 45°$$

$\therefore$ $$\angle FCE = 45°$$

[$\because$ In ΔEFC, $\angle EFC = 45°$; $\angle FEC = 90°$ $\therefore$ $\angle FCE = 45°$]

In ΔOCD, $\angle DCO = 45°$, $\therefore$ $\cos 45° = \frac{OD}{OC}$

$\therefore$ $$OC = \frac{OD}{\cos 45°} = \frac{50}{\cos 45°} = 70.71 \text{ mm}$$

Substituting the values of OC, OB and T_2 in equation (*iii*),

$$P \times 600 = 3536.77 \times 70.71$$

$\therefore$ $$P = \frac{3536.77 \times 70.71}{600} = \mathbf{416.8\ N.}\quad \textbf{Ans.}$$

(*ii*) *Width of band*

Given :

Max. tensile stress = 50 N/mm^2. Thickness, t = 3 mm.

Let b = Width of band

The maximum tension in the band is T_1. The value of T_1 is 11494.5 N

$\therefore$ Maximum tension = 11494.5 N

But maximum tension = Max. tensile stress × Area of cross-section of band

or $$11494.5 = 50 \times [b \times t] = 50 \times [b \times 3]$$

$\therefore$ $$b = \frac{11494.5}{50 \times 3} = \mathbf{76.63\ mm.}\quad \textbf{Ans.}$$

(*b*) **Differential Band Brake.** In case of differential band brake no end of the band is connected to the fulcrum. Fig. 8.13 shows a differential band brake in which the ends of the band are connected at A and B which are on different sides of the fulcrum O.

When the drum rotates in the clock-wise direction, the end of the band attached at A will be tight with tension T_1 whereas the end of the band attached at B will be slack with tension T_2 as shown in Fig. 8.13 (*a*). The value of force P at the end of the lever, can be obtained as explained below :

Let a = perpendicular distance from fulcrum O on the line of action of tension T_2

b = perpendicular distance from O on line of action of tension T_1

L = length of lever from O.

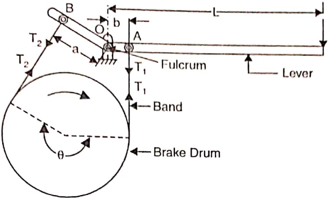
Consider the equilibrium of lever BOC. The force acting on the lever BOC are (*i*) force, P, (*ii*) tension T_1 at A and (*iii*) tension T_2 at B as shown in Fig. 8.13 (*b*).

Taking moments of all forces about O, we get

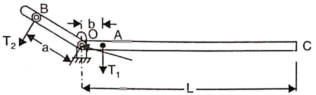
$$P \times L + T_1 \times b = T_2 \times a$$

or $$P \times L = T_2 \times a - T_1 \times b$$

$\therefore$ $$P = \frac{T_2 \times a - T_1 \times b}{L} \quad ...(i)$$



(*a*) Clockwise rotation.



(*b*) Equilibrium of lever *BOC*

Fig. 8.13. Differential band brake.

But T_1 is always more than T_2. Hence in the above equation the force P will be positive if

$$T_2 \times a > T_1 \times b$$

or $$T_1 \times b < T_2 \times a$$

or $$\frac{T_1}{T_2} < \frac{a}{b} \qquad ...(ii)$$

In equation (i), the force P will be zero or negative if

$$T_2 \times a \le T_1 \times b$$

or $$\frac{T_2}{T_1} \le \frac{b}{a} \qquad ...(iii)$$

If the force P is zero or negative, then the brake becomes as self locking. Hence for self-locking of the brake when drum rotates clock-wise the condition is

$$\frac{T_2}{T_1} \le \frac{b}{a} \qquad ...(8.12)$$

Anti-clockwise rotation

Fig. 8.14 shows a differential band brake in which brake drum is rotating anti-clock-wise. The end of the band connected to B will be tight with tension T_1 whereas the end of the band connecting to A will be slack with tension. T_2 .

Taking the moments about the fulcrum O, we get

$$P \times L + T_2 \times b = T_1 \times a$$

or $$P \times L = T_1 \times a - T_2 \times b$$

or $$P = \frac{T_1 \times a - T_2 \times b}{L}$$

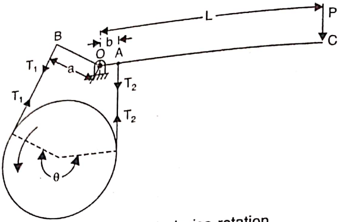


Fig. 8.14. Anti-clockwise rotation.

For self-locking of the brake, the force P should be zero or negative. But force P will be zero or negative if

$$T_1 \times a \leq T_2 \times b$$

or
$$\frac{T_1}{T_2} \leq \frac{b}{a} \qquad \text{...(8.13)}$$

Problem 8.9. *Fig. 8.15 shows a differential band brake of drum diameter 400 mm. The two ends of the band are fixed to the points on the opposite side of fulcrum of the lever at a distance of 50 mm and 160 mm from the fulcrum as shown in Fig. 8.15. The brake is to sustain a torque of 300 Nm. The co-efficient of friction between band and the brake is 0.2. The angle of contact is 210° and the length of lever from the fulcrum is 600 mm. Determine :*

(i) the force required at the end of the lever for the clockwise and anti-clockwise rotation of the drum.

(ii) value of OB for the brake to be self-locking for clockwise rotation.

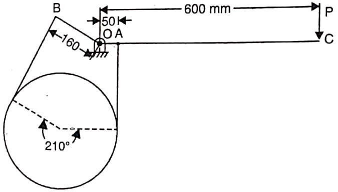


Fig. 8.15

Sol. Given :

Differential band brake, which means no ends of the band is connected to the fulcrum. The other data is :

d = 400 mm or r = 200 mm = 0.2 m ;

Distance OA = 50 mm, distance OB = 160 mm, T_B = 300 Nm, μ = 0.2, L = 600 mm,

$$\theta = 210° = 210 \times \frac{\pi}{180} = 3.665 \text{ rad.}$$

(i) Force at the end of lever for clock-wise rotation

Refer to Fig. 8.16. For the clockwise rotation of the drum, the end of the band connected to A will be tight with tension T_1 whereas the end of the band connected to B will be slack with tension T_2. Consider the equilibrium of BOC.

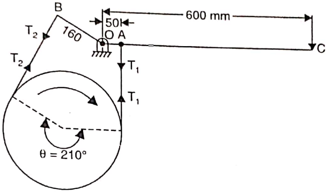


Fig. 8.16

Taking the moments of all forces acting on BOC,
(The focus acting on BOC are T_2, T_1 and P) about fulcrum O, we get

$$T_1 \times AO + P \times OC = T_2 \times OB \quad ...(i)$$

or

$$T_1 \times 50 + P \times 600 = T_2 \times 160 \quad ...(ii)$$

Let us first find the values of T_1 and T_2, so that the value of P can be obtained.
Using equation,

$$\frac{T_1}{T_2} = e^{\mu \times \theta}$$

or

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu \times \theta = 0.2 \times 3.665 = 0.733$$

or

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.733}{2.3} = 0.3187$$

$\therefore$

$$\frac{T_1}{T_2} = \text{Anti-log of } 0.3187 = 2.083$$

$\therefore$

$$T_1 = 2.083T_2 \quad ...(iii)$$

The braking torque is given by,

$$T_B = (T_1 - T_2) \times r$$

or

$$300 = (T_1 - T_2) \times 0.2 \quad (\because r = 0.2 \text{ m})$$

$$= (2.083T_2 - T_2) \times 0.2 \quad (\because T_1 = 2.083T_2)$$

$$= 1.083 \times T_2 \times 0.2$$

$\therefore$

$$T_2 = \frac{300}{1.083 \times 0.2} = 1385 \text{ N}$$

Substituting the value of T_2 in equation (*iii*), we get

$\therefore$

$$T_1 = 2.083 \times 1385 = 2884.95 \text{ N}$$

Substituting the values of T_1 and T_2 in equation (*ii*), we get

$$2884.95 \times 50 + P \times 600 = 1385 \times 160$$

or

$$144247.5 + 600P = 221600$$

$$600P = 221600 - 144247.5 = 77352.5$$

or

$$P = \frac{77352.5}{600} = \mathbf{128.92\ N.}\quad \textbf{Ans.}$$

or

Force at the end of the lever for anti-clockwise rotation

Refer to Fig. 8.17. For the anti-clockwise rotation of the drum, the end of the band attached to B will be tight with tension T_1 whereas the end attached to A will be slack with tension T_2. Consider the equilibrium of lever BOC. Taking moments of all forces (*i.e.* T_1, T_2 and P) about fulcrum O, we get

$$P \times 600 + T_2 \times 50 = T_1 \times 160$$

or

$$P = \frac{T_1 \times 160 - T_2 \times 50}{600} \quad ...(A)$$

The value of P can be obtained if values of T_1 and T_2 are known. The values of T_1 and T_2 are obtained by using equation :

$$\frac{T_1}{T_2} = e^{\mu\theta} \quad \text{or} \quad 2.3 \log \frac{T_1}{T_2} = \mu \times \theta$$

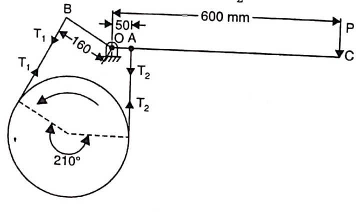


Fig. 8.17

or

$$\log\left(\frac{T_1}{T_2}\right) = \frac{\mu \times \theta}{2.3} = \frac{0.2 \times 3.665}{2.3} = 0.3187$$

or

$$\frac{T_1}{T_2} = \text{Antilog of } 0.3187 = 2.083$$

or

$$T_1 = 2.083T_2$$

The braking torque is given by equation,

$$T_B = (T_1 - T_2) \times r$$

or

$$300 = (2.083T_2 - T_2) \times 0.2 \qquad (\because\ T_1 = 2.083T_2 \text{ and } r = 0.2)$$

$$= 1.083T_2 - 0.2$$

$\therefore$

$$T_2 = \frac{300}{1.083 \times 0.2} = 1385 \text{ N}$$

and

$$T_1 = 2.083 \times T_2 = 2.083 \times 1385 = 2884.95 \text{ N}$$

Substituting the values of T_1 and T_2 in equation (A), we get

$$P = \frac{2884.95 \times 160 - 1385 \times 50}{600}$$

$$= \frac{461592 - 69250}{600} = \mathbf{653.9\ N.}\quad \textbf{Ans.}$$

(ii) Value of OB for the brake to be self-locking for clockwise rotation

Refer to Fig. 8.16. For clockwise rotation of the drum, we have equation (*i*), as

$$T_1 \times AO + P \times OC = T_2 \times OB$$

The brake will be self-locking, if P is zero. Hence substituting $P = 0$ in the above equation, we get

$$T_1 \times OA = T_2 \times OB$$

The values of T_1, T_2 and OA are known, hence the value of OB can be obtained

$$\therefore \quad OB = \frac{T_1 \times OA}{T_2}$$

$$= \frac{2884.95 \times 50}{1385} = \textbf{104.15 mm.} \quad \textbf{Ans.}$$

Problem 8.10. *Fig. 8.18 shows a barrel and a differential band brake which are keyed to the same shaft. A rope is wound round a barrel and supports a load of 200 kN. The brake drum diameter is 600 mm and diameter of barrel is 300 mm. The two ends of the bands are attached to pins on opposite sides of the fulcrum of the brake lever at a distances of 20 mm and 80 mm as shown in Fig. 8.18. The angle of contact of band brake is 270° and co-efficient of friction 0.25. Determine the minimum force required at the end of the lever to support the load, if the length of the lever from the fulcrum is 2400 mm.*

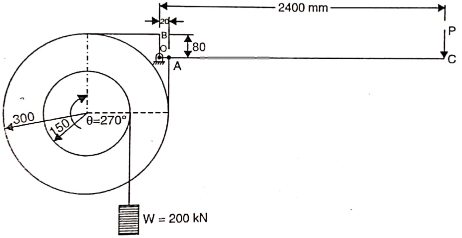


Fig. 8.18

Sol. Given :

Max. load, $W = 200$ kN $= 200 \times 10^3$ N ; $d = 600$ mm or $r = 300$ mm $= 0.3$ m ;

Dia. of barrel, $D = 300$ mm or radius of barrel, $R = 150$ mm $= 0.15$ m ;

Distance $OA = 20$ mm ; distance $OB = 80$ mm ; $\theta = 270° = 270 \times \frac{\pi}{180} = 4.712$ rad ;

$$\mu = 0.25, L = 2400 \text{ mm}$$

Let P = Minimum force at the end of the lever to support the load

As OB is greater than OA, therefore the force P will act downward.

Let us find the two values of P when drum rotates clockwise and then anti-clockwise.

(i) Drum rotates clockwise

Refer to Fig. 8.19. For clockwise rotation, the end of the band attached to A will be tight with tension T_1 whereas the end attached to B will be slack with tension T_2.

$\therefore \quad T_1 - T_2 = 10^5$

or $\quad (3.25T_2 - T_2) = 10^5 \quad [\because$ From (ii), $T_1 = 3.25T_2]$

or $\quad 2.25T_2 = 10^5$

$$\therefore \quad T_2 = \frac{10^5}{2.25} = 44444.44 \text{ N}$$

$$\therefore \quad T_1 = 3.25T_2 = 3.25 \times 44444.44$$

$$= 144444.44 \text{ N}$$

Substituting the value of T_2 in equation (i), we get

$$\therefore \quad P = \frac{44444.44}{30}$$

$$= \mathbf{1481.48\ N.} \quad \textbf{Ans.}$$

8.2.3. Band and Block Brake. If the band and also the blocks are used for applying brakes to a rotating body, then it is known as band and blockbrake. Fig. 8.21 shows the band and block brake which is the modification of the band brake and consists of a number of wooden blocks fixed inside a flexible steel band. The friction between the blocks and the drum provides braking action. When the brake is applied, the blocks are pressed against the drum. The wooden blocks have a higher co-efficient of friction. This increases the effectiveness of the brake. Also the wooden blocks can be easily replaced if worn out.

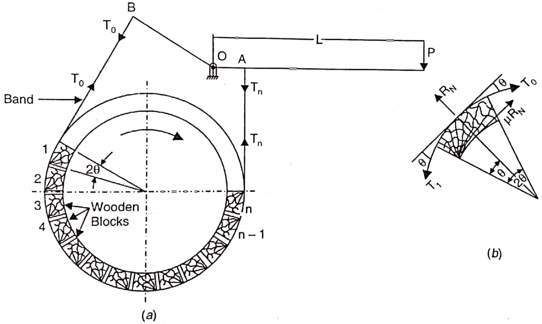


Fig. 8.21

Let there are 'n' number of blocks, each subtending on angle 2θ at the drum centre as shown in Fig. 8.21 (b). Let the drum rotates in clockwise direction. For clockwise rotation, the end of band attached to A will be tight with tension T_n whereas the end attached to B will be slack with tension, T_0.

Let n = Number of blocks

T_n = Tension on tight side after n block

T_0 = Tension on slack side

μ = Co-efficient of friction

T_1 = Tension in the band between first and second block

T_2 = Tension in the band between second and third block.

Consider one of the blocks (say first block) as shown in Fig. 8.21 (*b*). Each block embrass a short arc on the drum. The first block is in equilibrium under the action of following forces :

(*i*) Tension T_0 on the slack side

(*ii*) Tension T_1 on the tight side *i.e.* tension in the band between first and second block

(*iii*) Normal reaction R_N

(*iv*) Force of friction ($\mu \times R_N$) acting on the block in the direction of rotation of drum.

The frictional force on the drum will be in the opposite direction of the rotation of the drum (*i.e.* in anti-clockwise direction). And the frictional force on the *block* will be in the opposite direction of the frictional force on drum. Hence the friction force on the block will be in clockwise direction. This means that frictional force on the block will act in the direction of rotation of the drum.

Resolving the forces [acting on the block shown in Fig. 8.21 (*b*)] tangentially

$$T_1 \cos\theta - T_0 \cos\theta = \mu R_N \qquad (\because \; T_1 \text{ is more than } T_0)$$

or
$$(T_1 - T_0)\cos\theta = \mu R_N \qquad ...(i)$$

Resolving the force, radially

$$T_1 \sin\theta + T_0 \sin\theta = R_N$$

or
$$(T_1 + T_0)\sin\theta = R_N \qquad ...(ii)$$

Dividing equation (*i*) by equation (*ii*),

$$\frac{(T_1 - T_0)\cos\theta}{(T_1 + T_0)\sin\theta} = \frac{\mu R_N}{R_N}$$

or
$$\frac{(T_1 - T_0)}{(T_1 + T_0)} \times \frac{1}{\tan\theta} = \mu$$

or
$$\frac{(T_1 - T_0)}{(T_1 + T_0)} = \frac{\mu \tan\theta}{1}$$

Let us find the ratio of tensions T_1/T_0. From the above equation, we have

$$(T_1 - T_0) = (T_1 + T_0)\,\mu \tan\theta$$
$$= T_1 \times \mu \tan\theta + T_0 \times \mu \tan\theta$$

or
$$T_1 - T_1 \times \mu \tan\theta = T_0 + T_0\,\mu \tan\theta$$

or
$$T_1(1 - \mu \tan\theta) = T_0(1 + \mu \tan\theta)$$

or
$$\frac{T_1}{T_0} = \frac{1 + \mu \tan\theta}{1 - \mu \tan\theta}$$

Similarly, for the second block the ratio of tensions will be given by

$$\frac{T_2}{T_1} = \frac{1 + \mu \tan\theta}{1 - \mu \tan\theta} \text{ and so on.}$$

For the '*n*th' block, the ratio of tensions will be

$$\frac{T_n}{T_{n-1}} = \frac{1 + \mu \tan\theta}{1 - \mu \tan\theta}$$

Hence the ratio of tensions in the tight and slack sides of the complete band and block brake can be obtained as :

$$\frac{T_n}{T_0} = \frac{T_n}{T_{n-1}} \times \ldots\ldots \times \frac{T_3}{T_2} \times \frac{T_2}{T_1} \times \frac{T_1}{T_0}$$

$$= \left(\frac{1+\mu \tan \theta}{1-\mu \tan \theta}\right) \times \ldots\ldots \times \left(\frac{1+\mu \tan \theta}{1-\mu \tan \theta}\right) \times \left(\frac{1+\mu \tan \theta}{1-\mu \tan \theta}\right) \times \left(\frac{1+\mu \tan \theta}{1-\mu \tan \theta}\right)$$

$$= \left(\frac{1+\mu \tan \theta}{1-\mu \tan \theta}\right)^n \qquad \ldots(8.14)$$

where θ = Half of the angle subtended by each block on the centre of the drum.

Net torque on drum $= (T_n - T_0) \times r$

Now the braking torque on the drum will be

$$T_B = (T_n - T_0) \times r \qquad \ldots(8.15)$$

where r = Effective radius of band.

Problem 8.12. *The maximum braking torque acting on a band and block brake (shown in Fig. 8.22 (a)) is 2000 Nm. The band is lined with 15 blocks each of which subtends an angle of 12° at the centre of rotating drum. The co-efficient of friction between the band and block is 0.3. The diameter of the drum is 680 mm whereas the thickness of blocks is 60 mm. Find the least force required at the end of the lever which is 480 mm long.*

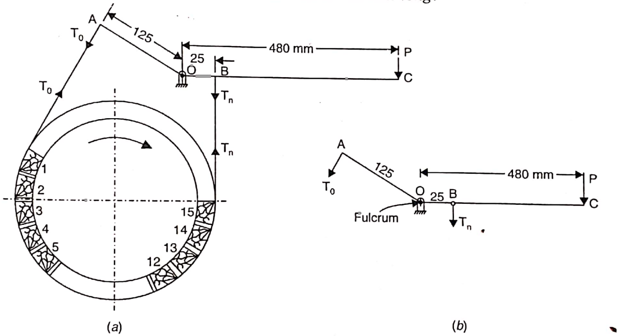


Fig. 8.22

Sol. Given :

T_B = 2000 Nm ; n = 15 ; 2θ = 12° or θ = 6° = $6 \times \frac{\pi}{180}$ = 0.1047 rad ;

μ = 0.3, d = 680 mm ; t = 60 mm ;

Dia. of band = $d + 2t$ = 680 + 2 × 60 = 800 mm or radius of band r = 400 mm = 0.4 m ;

L = 480 mm

As distance OA > distance OB, hence force P must be applied at C downwards.

The force P will be least, if the end of the band attached to A is slack and the end attached to B is tight.This is only possible if drum rotates clockwise as shown in Fig. 8.22 (a).

For the band and block brake, using equation (8.14), we get

$$\frac{T_n}{T_0} = \left(\frac{1+\mu \tan \theta}{1-\mu \tan \theta}\right)^n$$

$$= \left(\frac{1+0.3\times\tan 6^\circ}{1-0.3\times\tan 6^\circ}\right)^{15} = \left(\frac{1+0.3\times 0.1051}{1-0.3\times 0.1051}\right)^{15}$$

$$= \left(\frac{1.0315}{0.9685}\right)^{15} = (1.065)^{15} = 2.573$$

$\therefore$ $T_n = 2.573 T_o$ ...(*i*)

The braking torque is given by equation (8.15),

$$T_B = (T_n - T_0) \times r$$

or $2000 = (2.573T_o - T_o) \times 0.4$ $\quad (\because \; T_n = 2.573T_0 \text{ and } r = 0.4)$

$$= 1.573 T_o \times 0.4$$

or $T_o = \dfrac{2000}{1.573 \times 0.4} = 3178.6$

$\therefore$ $T_n = 2.573 \times T_o = 2.573 \times 3178.6 = 8178.5$ N

Now taking the moments of all force acting on the brake lever, about fulcrum, we get [Refer to Fig. 8.22 (*b*)]

$$P \times 480 + T_n \times 25 = T_o \times 125$$

or $480P = T_o \times 125 - T_n \times 25$

or $P = \dfrac{T_0 \times 125 - T_n \times 25}{480}$

$$= \frac{3178.6 \times 125 - 8178.6 \times 25}{480}$$

$$= \frac{4767900 - 204465}{480} = \textbf{401.8 N.} \quad \textbf{Ans.}$$

Problem 8.13. *Find :*

(i) the maximum braking torque,

(ii) the angular retardation of the brake drum and

(iii) the time taken by the system to come to rest from the rated speed of 240 r.p.m.

When a band and block having 12 blocks, each of which subtends and angle of 18° at the drum centre, is applied to a rotating drum of diameter 800 mm. The blocks are 100 mm thick. The drum and the flywheel mounted on the same shaft have a mass of 1600 kg and have a combined radius of gyration of 500 mm. The two ends of the band are attached to the pins on the opposite sides of the brake fulcrum at a distance of 35 mm and 140 mm from the fulcrum. The co-efficient of friction between the blocks and drum may be taken as 0.3. A force of 150 N is applied at a distance of 800 mm from the fulcrum to apply the brake.

Sol. Given :

$n = 12$; $2\theta = 18°$ or $\theta = 9°$; dia. of drum, $d = 800$ mm ; thickness of block, $t = 100$ mm, total mass, $m = 1600$ kg ; combined radius of gyration, $k = 500$ mm $= 0.5$ m ; $OB = 35$ mm ; $OA = 140$ mm ; $\mu = 0.3$; $L = 800$ mm, $P = 150$ N

Dia. of band, $D = d + 2t = 800 + 2 \times 100 = 1000$ mm ; $r = 500$ mm $= 0.5$ m

(i) Maximum Braking Torque

As distance *OA* is greater than distance *OB*, the force *P* must act downwards at *C*. For maximum braking torque (or for least force to apply the brake), the brake should be arranged so that the tight side of the band is attached to the shorter distance *i.e.* tight side of band should be attached to *B*. This is possible if drum rotates clockwise, as shown in Fig. 8.23.

∴ Braking torque is given by equation (8.15) as

$$T_B = (T_n - T_o) \times r$$
$$= (12335 - 3940.88) \times 0.5 = \mathbf{4197\ Nm.}\quad \textbf{Ans.}$$

(Dia. of band $= d + 2t = 800 + 2 \times 100 = 1000$ m

$\therefore\ r = \dfrac{1000}{2} = 500 \text{ mm} = 0.5 \text{ m}$)

(ii) Angular retardation of the brake drum

Let α = Angular retardation

We know that

Net torque $= I \times \alpha$

(where I = mass moment of inertia $= m \times k^2 = 1600 \times 0.5^2 = 400$)

or $(T_n - T_o) \times r = I \times \alpha$

or Braking torque $= I \times \alpha$

$\therefore$ $4197 = mk^2 \times \alpha$

$= 1600 \times 0.5^2 \times \alpha = 400 \times \alpha$

$\therefore$ $\alpha = \dfrac{4197}{400} = \mathbf{10.49\ rad/s^2.}$ **Ans.**

(iii) Time taken by the system to come to rest from the rated speed of 240 r.p.m.

Let t = Time taken

Initial angular speed, $\omega_0 = \dfrac{2\pi N_0}{60} = \dfrac{2\pi \times 240}{60} = 25.13$ rad/s

Final angular speed, $\omega = 0$

Using $\omega = \omega_0 - \alpha \times t$ (–ve sign is due to retardation)

$\therefore$ $t = \dfrac{\omega_0 - \omega}{\alpha}$

$= \dfrac{25.13 - 0}{10.49} = \mathbf{2.39\ s.}$ **Ans.**

8.2.4. Internal Expanding Shoe Brake. If the shoe is provided on the interior of the rotating drum and braking effect is produced due to the expansion of the shoe, then it is known as internal expanding shoe brake. Fig. 8.24 shows an internal expanding shoe brake which are commonly used in motor cars and light trucks. This consists of two shoes S_1 and S_2 whose outer surfaces are lined with some friction material (generally with Ferodo) to increase the co-efficient of friction. Each shoe is pivoted at one end about a fixed fulcrum O_1 and O_2 and the other end is having contact with a cam (or with each piston in a common hydraulic cylinder. There are two equal diameter pistons in a common hydraulic cylinder.) As the cam rotates, the shoes are pushed outwards and makes a contact with the drum. The friction between shoes and drum produces the braking torque and hence reduces the speed of the drum. The shoes are held in off position by a spring as shown in Fig. 8.24. Under normal running of the vehicle the drum rotates freely as the outer diameter of the shoes is a little less than the internal diameter of the drum.

The force required to operate such a brake can be calculated if we know the total forces acting on such a brake. Let us consider, the drum is rotating in anti-clockwise direction. For anti-clockwise rotation of the drum, the left hand shoe is known as *leading or primary shoe* whereas the right hand shoe is known as *trailing or secondary shoe*.

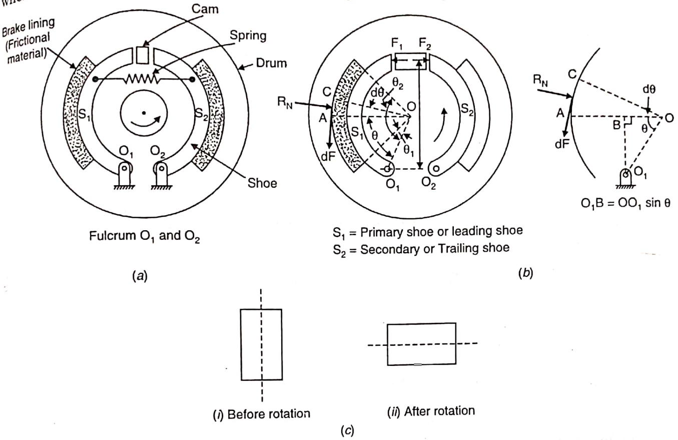


Fig. 8.24

Let F_1 = Force exerted by cam on the leading shoe

F_2 = Force exerted by cam on the trailing shoe

b = Width of brake lining

r = Internal radius of the wheel drum

p_n = Normal pressure

p_1 = Maximum intensity of normal pressure for the leading shoe

p_2 = Maximum intensity of normal pressure for trailing shoe

Consider a small length of brake lining say length AC which subtends an angle $\delta\theta$ at the centre as shown in Fig. 8.24 (*b*). Also let length OA makes an angle θ with OO_1. As the shoe turns about O_1, the rate of wear of the shoe lining at A will be directly proportional to the perpendicular distance from O_1 to OA *i.e.* distance O_1B as shown in Fig. 8.24 (*c*).

Now from the figure, we have

$$O_1B = OO_1 \sin\theta$$

Hence rate of wear at $A \propto O_1B$ or $OO_1 \sin\theta$ or $\sin\theta$

The normal pressure at A (*i.e.* p_N) is written as

$$p_N \propto \sin\theta$$

or

$$p_N = p_1 \sin\theta$$

where p_1 is constant of proportionality and is known as maximum intensity of pressure

∴ Normal force acting on the element,

$$\delta R_N = \text{Normal pressure} \times \text{Area of element}$$
$$= p_N \times (b \times r \,.\, \delta\theta)$$
$$= p_1 \times \sin\theta \times b \times r\,\delta\theta$$

∴ Braking or friction force on the element,

$$dF = \mu \times \delta R_N$$
$$= \mu \times p_1 \times \sin\theta \times br\,\delta\theta$$

∴ Braking torque, due to element about O,

$$\delta T_B = \delta F \times r$$
$$= \mu \times p_1 \times \sin\theta \times b \times r \times \delta\theta \times r$$
$$= \mu p_1 \times br^2 \times \sin\theta \,.\, \delta\theta$$

The total braking torque about O for whole of one shoe (*i.e.* leading shoe) is obtained by integrating the above equation between limits θ_1 and θ_2.

$$\therefore \quad T_{B_1} = \int_{\theta_1}^{\theta_2} \mu p_1 \times br^2 \times \sin\theta \times d\theta$$

$$= \mu \times p_1 \times br^2 \times \int_{\theta_1}^{\theta_2} \sin\theta \,.\, d\theta$$

$$= \mu \times p_1 \times br^2 \times \Big[-\cos\theta\Big]_{\theta_1}^{\theta_2}$$

$$= \mu p_1 br^2 \,[\cos\theta_1 - \cos\theta_2] \qquad \text{...(8.16)}$$

Moment of normal force (δR_N) about fulcrum O_1,

$$\delta M_N = \delta R_N \times O_1B = \delta R_N \times (OO_1 \sin\theta)$$
$$= (p_1 \times \sin\theta \times b \times r \times \delta\theta) \times (OO_1 \sin\theta)$$
$$= p_1 \times \sin^2\theta \times (b \times r \,.\, \delta\theta) \times OO_1$$

Total moment of normal forces for the leading shoe about O_1,

$$M_{N_1} = \int_{\theta_1}^{\theta_2} p_1 \times \sin^2\theta \times (b \times r \times \delta\theta)\, OO_1$$

$$= p_1 \times b \,.\, r \,.\, OO_1 \int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta = p_1 \times b \times r$$

$$\times\, OO_1 \int_{\theta_1}^{\theta_2} \left[\frac{1 - \cos 2\theta}{2}\right] d\theta \qquad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= p_1 \times b \times r \times OO_1 \left[\frac{\theta - \dfrac{\sin 2\theta}{2}}{2}\right]_{\theta_1}^{\theta_2} = \frac{1}{2} \times p_1 \times b \times r \times OO_1 \left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{2} \times p_1 \times b \times r \times OO_1 \left[\theta_2 - \frac{\sin 2\theta_2}{2} - \left(\theta_1 - \frac{\sin 2\theta_1}{2}\right)\right]$$

$$= \frac{1}{2} \times p_1 \times b \times r \times OO_1 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2)\right] \qquad \text{...(8.17)}$$

Moment of frictional force δF about the fulcrum O_1

$$\delta M_F = \delta F \times AB = \delta F\,[AO - OB] = \delta F\,[r - OO_1 \cos\theta]$$

$$= \mu p_1 \sin\theta \times br\,\delta\theta\,[r - OO_1\cos\theta] \qquad [\because \delta F = \mu p_1 \sin\theta \times b \times r \times \delta\theta]$$

$$= \mu p_1 \times br\,[r\sin\theta - OO_1 \cos\theta \sin\theta]\,\delta\theta$$

$$= \mu p_1 \times br\left[r\sin\theta - \frac{OO_1}{2}\sin 2\theta\right]\delta\theta \qquad [\because 2\cos\theta\sin\theta = \sin 2\theta]$$

Total moment of frictional force about fulcrum O_1 for one shoe is obtained by integrating the above equation between limits θ_1 and θ_2.

$$\therefore \text{ Total moments, } \quad M_{F_1} = \int_{\theta_1}^{\theta_2} \mu p_1\, br\left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta\right)\delta\theta$$

$$= \mu p_1 \times b \times r \int_{\theta_1}^{\theta_2}\left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta\right)\delta\theta$$

$$= \mu p_1 \times b \times r\left[r(-\cos\theta) - \frac{OO_1}{2}\left(-\frac{\cos 2\theta}{2}\right)\right]_{\theta_1}^{\theta_2}$$

$$= \mu p_1 \times b \times r\left[-r\cos\theta + \frac{OO_1}{4}\cos 2\theta\right]_{\theta_1}^{\theta_2}$$

$$= \mu p_1 \times b \times r\left[-r\cos\theta_2 + \frac{OO_1}{4}\cos 2\theta_2 - \left(-r\cos\theta_1 + \frac{OO_1}{4}\cos 2\theta_1\right)\right]$$

$$= \mu p_1 \times b \times r\left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_1}{4}(\cos 2\theta_2 - \cos 2\theta_1)\right] \qquad \text{...(8.18)}$$

Now the force required to operate the leading or primary shoe is obtained by taking moments of all forces acting on the leading shoe about fulcrum O_1. [See Fig. 8.24 (*c*)].

Taking moments about O_1, we get

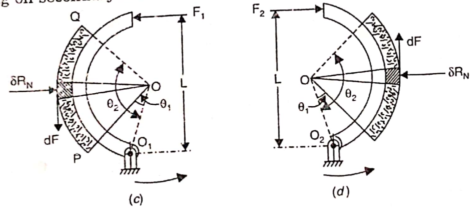
$F_1 \times L$ + Total moment due to normal force + Total moment due to frictional force = 0

or $\quad F_1 \times L - M_{N_1} + M_{F_1} = 0 \qquad$ (Moment due to R_N is clockwise, whereas due to F_1 and frictional force it is anti-clockwise)

...(8.19)

or $\quad F_1 \times L = M_{N_1} - M_{F_1}$

The force (F_2) required to operate the secondary shoe is obtained by taking moments of all forces acting on secondary shoe, about fulcrum O_2. [See Fig. 8.24 (*d*)].



(*c*) (*d*)

Fig. 8.24

$$\therefore \quad F_2 \times L = M_{N_2} + M_{F_2} \quad \text{...(8.20)}$$

If $M_{F_2} > M_{N_2}$, the brake will be self-locking.

where $$M_{N_2} = \frac{1}{2} \times p_2 \times b \times r \times OO_2 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2)\right]$$

$$M_{F_2} = \mu \times p_2 \times b \times r \left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_2}{4}(\cos 2\theta_2 - \cos 2\theta_1)\right]$$

and $$T_{B_2} = \mu \times p_2 \times b \times r^2 [\cos\theta_1 - \cos\theta_2] \quad \text{...(8.21)}$$

and total braking torque on both shoes,

$$T_B = T_{B_1} + T_{B_2} \quad \text{...(8.22)}$$

$$= \mu \times p_1 \times b \times r^2 [\cos\theta_1 - \cos\theta_2] + \mu \times p_2 \times b \times r^2 [\cos\theta_1 - \cos\theta_2]$$

$$= \mu \times b \times r^2 \times [\cos\theta_1 - \cos\theta_2](p_1 + p_2) \quad \text{...(8.23)}$$

Problem 8.14. *Calculate the braking torque applied by an internal expanding shoe brake shown in Fig. 8.24 on the rotating drum of diameter 300 mm if the drum is rotating (i) anti-clockwise and (ii) clockwise. The other data given is :*

Force F on each shoe	$= 90$ *N*
Co-efficient of friction,	$\mu = 0.3$
Width of the brake lining,	$b = 40$ *mm*
Angles	$\theta_1 = 30°,\ \theta_2 = 135°$
Distance :	$L = 200$ *mm*, $OO_1 = 120$ *mm.*

Sol. Given : [Refer to Fig. 8.24]

Dia. of drum, $d = 300$ mm or radius $r = 150$ mm $= 0.15$ m ; $F_1 = F_2 = 90$ N ; $\mu = 0.3$; $b = 40$ mm $= 0.04$ m ; $\theta_1 = 30°$; $\theta_2 = 135°$; $L = 200$ mm $= 0.2$ m ; $OO_1 = 120$ mm $= 0.12$ m.

(i) Braking Torque for anti-clockwise rotation

Let T_B = Total braking torque

$$= T_{B_1} + T_{B_2}$$

where T_{B_1} = braking torque on leading shoe

T_{B_2} = braking torque on trailing shoe

The braking torque on leading shoe is given equation (8.16) as

$$T_{B_1} = \mu \times p_1 \times b \times r^2 (\cos\theta_1 - \cos\theta_2) \quad \text{...}(i)$$

In the above equation, the value of p_1 is unknown. Let us first find the value of p_1. This is obtained by using equation (8.19) as

$$F_1 \times L = M_{N_1} - M_{F_1}$$

$$= \frac{1}{2} \times p_1 \times b \times r \times OO_1 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2)\right]$$

$$- \mu \times p_1 \times br \times \left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_1}{4}(\cos 2\theta_2 - \cos 2\theta_1)\right].$$

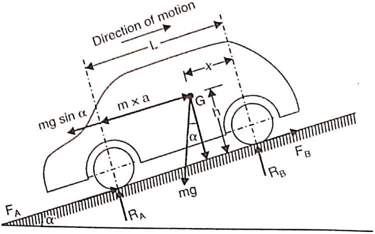


Fig. 8.29

Cancelling 'm' to both sides, we get

$$\mu \times g \cos \alpha - g \sin \alpha = a$$

∴ Retardation, $a = \mu \times g \times \cos \alpha - g \sin \alpha$

$$= g(\mu \cos \alpha - \sin \alpha)$$

$$= 9.81(0.6 \times \cos 15° - \sin 15°)$$

$$= 9.81(0.6 \times 0.966 - 0.2588) = 3.147 \text{ m/s}^2$$

∴ Acceleration, $a = -3.147 \text{ m/s}^2$

Now using equation, $v^2 - u^2 = 2 \times a \times S$

or $0^2 - 15^2 = 2 \times (-3.147) \times S$

∴ $S = \dfrac{-15^2}{2 \times (-3.147)} = \mathbf{35.75 \text{ m.}}$ **Ans.**

The time 't' is obtained by using equation,

$$v = u + at$$

or $0 = 15 + (-3.147) \times t$

∴ $t = \dfrac{15}{3.147} = \mathbf{4.76 \text{ seconds.}}$ **Ans.**

8.4. DYNAMOMETER

A dynamometer is a device used to measure the frictional resistance or frictional torque. This frictional resistance or frictional torque is obtained by applying a brake. The dynamometer consists of a brake and also a device of measuring the braking force (or braking torque). Hence dynamometer is a brake with a device of measuring the frictional resistance or frictional torque. After knowing frictional torque, the power of the engine can be obtained.

Following are the two types of dynamometers :

(*i*) Absorption dynamometers and

(*ii*) Transmission dynamometer.

Absorption dynamometers absorb the available power in doing work against friction whereas transmission dynamometers transmit the available power to some other machines where the power is suitably measured.

8.5. ABSORPTION DYNAMOMETER

Absorption dynamometers consists of some form of brakes in which provision is made for measuring the frictional torque on the drum. The following are the important types of absorption dynamometer :

(*i*) Prony brake dynamometer

(*ii*) Rope brake dynamometer.

8.5.1. Prony Brake Dynamometer. Fig. 8.30 shows the Prony brake dynamometer which consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. Each of the wooden blocks embraces rather less than one half of the pulley rim. The two blocks can be drawn together by means of bolts, nuts and springs so as to increase the pressure on the pulley. The lower block carries an arm (lever) to the end of which a weight W can be applied. A second arm projects from the block in the opposite direction and carries a balance weight B, which balances the brake when unloaded. Two stops S, S are provided and the lever arm will float between these stops. The friction torque on the pulley may be increased by screwing up the bolts, until it balances the torque due to available power.

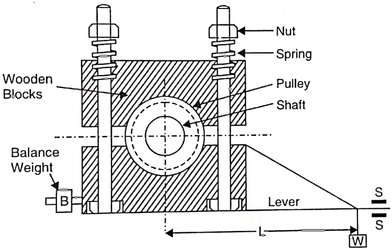


Fig. 8.30

For measuring the power of the engine, the long end of the lever is loaded with a known weight W. Now the nuts are tightened until the shaft runs at a constant speed and the lever is in horizontal position. Under these conditions, the torque due to weight W will balance the frictional torque on the pulley due to the frictional resistance between the blocks and the pulley. This means that the moment due to weight W will be equal to the frictional torque.

Let W = Weight at the end of the lever,

R = Radius of the pulley,

μ = Co-efficient of friction between pulley and blocks

L = Horizontal distance of weight W from the centre of the pulley,

N = Speed of the shaft in r.p.m.

Torque on the shaft, $T = W \times L$

∴ Power of the engine = Torque × Angular speed

$= T \times \omega$

$= T \times \dfrac{2\pi N}{60}$

$$= W \times L \times \frac{2\pi N}{60} \text{ Watts.} \qquad (\because \; T = W \times L)$$

From the above equation, it is clear that the power of the engine is independent of: (*i*) radius of the pulley, R (*ii*) co-efficient of friction between pulley and wooden blocks and (*iii*) pressure exerted by tightening the nuts.

8.5.2. Rope Brake Dynamometer. Fig. 8.31 shows the rope brake dynamometer which consists of one, two or more ropes wound round the rim of a pulley (or flywheel) fixed rigidly to the shaft of the engine whose power is required to be measured. The upper end of the ropes is attached to a spring balance (S) while the lower end carries the dead weight W. The ropes are spaced evenly across the width of the rim by means of three or four wooden blocks at different points round the rim (or around the circumference of the flywheel).

For measuring the power of an engine, the engine is made to run at a constant speed. Under this condition, the torque transmitted by the engine must be equal to the frictional torque due to the ropes.

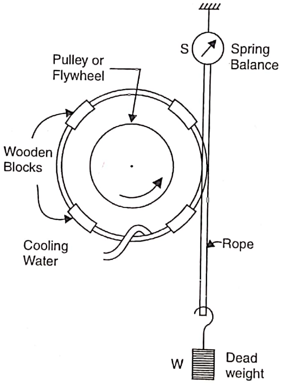


Fig. 8.31

Let N = Constant speed of the engine shaft

W = Deed weight

S = Spring balance reading

D = Diameter of flywheel (or Dia. of the rim of pulley)

d = Dia. of rope

Then net load on brake = $(W - S)$

$\therefore$ Frictional torque due to ropes

= (Net load on brake) × Distance of load line from the centre of shaft

$$= (W - S) \times \left(\frac{D + d}{2}\right)$$

But torque transmitted by engine at constant speed

$$= \text{Frictional torque due to ropes}$$

$$= (W - S) \times \left(\frac{D+d}{2}\right)$$

$\therefore$ Brake power of engine $= \text{Torque transmitted by engine} \times \text{Angular speed of engine}$

$$= (W - S) \times \left(\frac{D+d}{2}\right) \times \omega$$

$$= (W - S) \times \left(\frac{D+d}{2}\right) \times \frac{2\pi N}{60} \quad \left(\because \omega = \frac{2\pi N}{60}\right) \quad \text{...(8.27)}$$

If dia. of rope (*i.e.* d) is neglected, then brake power of engine

$$= (W - S) \times \frac{D}{2} \times \frac{2\pi N}{60}$$

$$= (W - S) \times R \times \frac{2\pi N}{60} \text{ Watts} \quad \left(\because R = \frac{D}{2}\right) \quad \text{...(8.28)}$$

A cooling arrangement is necessary if the brake power of the engine is very large, as in that case the heat produced due to friction between ropes and the flywheel will also be very large. For cooling the rim, the rim should be of channel section on the inside so that cold water may be supplied at one point, carried round the rim and then removed.

Problem 8.17. *Calculate the brake power of an engine which is running at a constant speed of 300 r.p.m. and carries a rope brake dynamometer. The dead weight on the engine and spring balance readings are 550 N and 100 N respectively. The diameters of flywheel and rope are 1.8 m and 18.75 mm respectively.*

Sol. Given :

$N = 300$ r.p.m. ; $W = 550$ N ; $S = 100$ N ; $D = 1.8$ m and $d = 18.75$ mm $= 0.01875$ m

Using equation (8.27) for brake power, we get

Brake power $= (W - S) \times \left(\frac{D+d}{2}\right) \times \frac{2\pi N}{60}$

$$= (550 - 100) \times \left(\frac{1.89 + 0.01875}{2}\right) \times \frac{2\pi \times 300}{60} \text{ Watts}$$

$$= 450 \times 0.909375 \times 10\pi$$

$$= 12856 \text{ Watts} = \textbf{12.856 kW}. \quad \textbf{Ans.}$$

8.6. TRANSMISSION DYNAMOMETER

In case of transmission dynamometer, the energy or power is not absorbed. Hence the energy is not wasted in friction but is used for doing work. The energy or power produced by the engine is transmitted through the dynamometer to some other machines where the power developed is suitably measured. The following are the important types of transmission dynamometer :

(*i*) Epi-cyclic train dynamometer,

(*ii*) Belt transmission dynamometer, and

(*iii*) Torsion dynamometer.

8.6.1. Epi-cyclic Train Dynamometer. Fig. 8.32 show an epi-cyclic train dynamometer which consists of a simple epi-cyclic train of gears *i.e.* spur gear A, internal wheel D having internal teeth and an intermediate wheel C (*i.e.* pinion). The wheel A with spur gear is keyed to the *driving shaft* (*i.e.* engine shaft). Let it rotates in anti-clockwise direction. The wheel D having internal teeth is keyed to the *driven shaft* and it will rotate in clockwise direction. The gears of the wheel C (which is the intermediate wheel and known as pinion) meshes both the gears of wheel A and of wheel D. Thus the power is transmitted from wheel A to wheel D through the intermediate wheel C. The wheel C revolves freely on a pin fixed to the arm of a lever. The lever is pivoted about the common axis of the driving and driven shafts *i.e.* at point E. When the dynamometer is at rest, the weight B balances the lever arm.

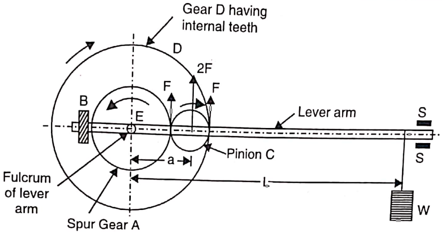


Fig. 8.32

When the dynamometer is in operation, the tangential force exerted by the wheel A on the wheel C and the tangential *reaction* of the wheel D on the wheel C will act in the upward direction. Also if the friction of the pin on which wheel C revolves is neglected, then the above two forces will be equal. Hence the total upward force on the lever arm acting through the axis of the wheel C is $2F$. This force tends to rotate the lever arm about its fulcrum E. The torque due to force $2F$ on the arm will be $2F \times a$. This torque will be balanced by the torque due to a dead weight W placed at the end of the lever as shown in Fig. 8.32. The lever arm floats between the stops S, S.

Hence for equilibrium of the lever arm,

Torque due to force $2F$ = Torque due to dead weight W

or $$2F \times a = W \times L$$

$$\therefore \quad F = \frac{W \times L}{2a}$$

Let R = Radius of wheel A

N = Speed of the rotation of driving shaft (*i.e.* speed of engine)

$\therefore$ Torque transmitted by engine

$$= F \times R = \frac{W \times L}{2a} \times R$$

$\therefore$ Power transmitted $= \text{Torque transmitted} \times \omega$

$$= \left(\frac{W \times L}{2a} \times R\right) \times \frac{2\pi N}{60} \text{ Watts.}$$

8.6.2. Belt Transmission Dynamometer. A belt transmission dynamometer measures the difference between the tensions on the tight and slack sides of a belt when it is running from one pulley to another pulley (*i.e.* when belt is transmitting power from one pulley to another pulley). This difference of tensions (*i.e.* $T_1 - T_2$) when multiplied by the speed of the belt, gives the power transmitted.

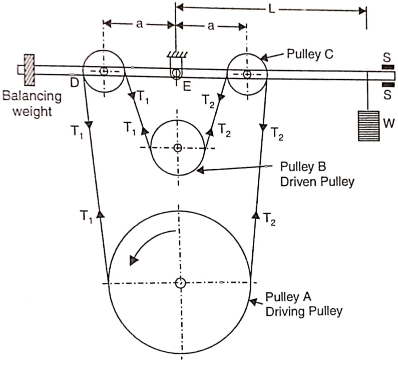


Fig. 8.33

Fig. 8.33 show a belt transmission dynamometer which is also called Tatham dynamometer. It consists of a driving pulley *A*, driven pulley *B* and intermediate pulleys *C* and *D*. The driving pulley *A* is rigidly fixed to the shaft of an engine whose power is to be measured. The driven pulley *B* is fixed to another shaft to which power is to be transmitted. The intermediate pulleys *C* and *D* rotates on pins fixed to the lever. The lever is pivoted at *E*, the mid point of the two intermediate pulley centres. A continuous belt runs over the driving and the driven pulleys through the two intermediate pulleys. The movement of lever is controlled between two stops *S* and *S* one on each side of the lever.

Let the driving pulley *A* rotates anti-clockwise. The tight and slack sides of the belt will be as shown in Fig. 8.33. The total downward force acting on pulley *D* is $2T_1$ whereas the total downward force on pulley *C* is $2T_2$. As $2T_1$ is greater than $2T_2$, therefore the lever starts rotating about *E* in anti-clockwise direction. In order to balance it, a weight *W* is suspended at a distance *L* from *E* on the lever as shown in Fig. 8.33.

When the lever is in horizontal position, the total moments of all the force about fulcrum *E* should be zero *i.e.*

Total anti-clockwise moment = Total clockwise moment

or $$2T_1 \times a = 2T_2 \times a + W \times L$$

or $$2T_1 \times a - 2T_2 \times a = W \times L$$

or $$2a(T_1 - T_2) = W \times L \quad \text{...(8.29)}$$

or $$(T_1 - T_2) = \frac{W \times L}{2a}$$

Let v = Belt speed in m/s
D = Dia. of pulley A
N = Speed of engine shaft

Then $v = \frac{\pi DN}{60}$

∴ Power of the engine $= (T_1 - T_2) \times v$

$$= (T_1 - T_2) \times \frac{\pi DN}{60} \text{ Watts} \quad ...(8.30)$$

Note. The power may also be transmitted through the dynamometer from pulley B to pulley A.

Problem 8.18. *The driving pulley in a belt transmission dynamometer shown in Fig. 8.33 rotates at 400 r.p.m. The diameter of the driving pulley is 750 mm, whereas the diameters of pulleys B, C and D are 250 mm each. The load W is suspended at a distance of 800 mm from the fulcrum E.*

Find : (i) the value of the weight W required to maintain the lever in a horizontal position when power transmitted is 8 kW.

(ii) the value of W, when the belt just begins to slip on the driving pulley A. The co-efficient of friction is 0.2 and the maximum tension in the belt is 1600 N.

Sol. Given : (Refer to Fig. 8.33)

$N_A = 400$ r.p.m. ; $D_A = 750$ mm $= 0.75$ m ; $D_B = D_C = D_D = 250$ mm $= 0.25$ m ;
$L = 800$ mm $= 0.8$ m ; Power $= 8$ kW $= 8 \times 10^3$ W $= 8000$ W.

(i) Value of W when power transmitted is 8 kW or 8000 W

$$a = \frac{D_B}{2} + \frac{D_C}{2} = \frac{0.25}{2} + \frac{0.25}{2} = 0.25 \text{ m}$$

Let T_1 = Tension on tight side of the belt on pulley A
T_2 = Tension on slack side of the belt on pulley A.

For power transmitted, using equation (8.30), we get

$$\text{Power} = (T_1 - T_2) \times \frac{\pi DN}{60}$$

or
$$8000 = (T_1 - T_2) \times \frac{\pi \times 0.75 \times 400}{60}$$

∴
$$(T_1 - T_2) = \frac{8000 \times 60}{\pi \times 0.75 \times 400} = 509.3$$

But from equation (8.0), we have

$$(T_1 - T_2) = \frac{W \times L}{2a}$$

or
$$509.3 = \frac{W \times 0.8}{2 \times 0.25}$$

∴
$$W = \frac{509.3 \times 2 \times 0.25}{0.8} = \mathbf{318.3 \text{ N.}} \quad \textbf{Ans.}$$

(ii) Value of W, when the belt just begins to slip on driving pulley A.

$\mu = 0.2$ and Max. tension *i.e.* $T_1 = 1600$ N. Also from Fig. 8.33, it is clear that lap angle $\theta = 180° = \pi$

We know that $$\frac{T_1}{T_2} = e^{\mu \times \theta}$$

or $$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu \times \theta = 0.2 \times \pi = 0.6284$$

or $$\log\left(\frac{T_1}{T_2}\right) = \frac{0.6284}{2.3} = 0.2732$$

or $$\frac{T_1}{T_2} = \text{Antilog of } 0.2732 = 1.876$$

or $$T_2 = \frac{T_1}{1.876} = \frac{1600}{1.876} = 852.87 \text{ N}$$

Using equation (8.29), we get

$$(T_1 - T_2) = \frac{W \times L}{2a}$$

or $$(1600 - 852.87) = \frac{W \times 0.8}{2 \times 0.25}$$

or $$747.13 = \frac{W \times 0.8}{0.5}$$

or $$W = \frac{747.13 \times 0.5}{0.8} = \mathbf{466.95 \text{ N.}} \quad \textbf{Ans.}$$

8.6.3. Torsion Dynamometer. The torsion dynamometer works on the principle of angle of twist in a shaft when power is transmitted along the shaft. Actually the torque transmitted is directly proportional to the angle of twist. Hence if angle of twist can be measured accurately, then the corresponding torque transmitted can be calculated.

The driving end of a shaft twists through a small angle relative to the driven end when power is transmitted along the shaft. The angle of twist is obtained from the torsion equation which is given below as

$$\frac{T}{J} = \frac{C \times \theta}{L}$$

where T = Torque transmitted,

J = Polar moment of inertia of shaft,

θ = Angle of twist in radians,

L = Length of shaft,

C = Modulus of rigidity of the shaft material.

From the equation, we have

$$T = \frac{C \times \theta}{L} \times J \text{ where } \quad J = \frac{\pi}{32} \times D^4 \quad \text{... For a solid shaft}$$

$$= \frac{\pi}{32}(D_0^4 - D_i^4) \quad \text{... For a hollow shaft}$$

For a given shaft, the values of C, J and L are constant. Hence

$$T = k \times \theta \qquad \text{where } k = \frac{C \times J}{L} \text{ is a constant}$$

or $$\text{Torque transmitted} \propto \theta$$

Hence torque transmitted is directly proportional to the angle of twist. If angle of twist can be measured by some means then torque can be calculated. From the torque, the power transmitted can be obtained.

In actual practice, the angle of twist is measure for a small length of the shaft, therefore some magnifying device must be incorporated in the dynamometer for accurate measurement of the angle of twist. The Bevis-Gibson flash light torsion dynamometer uses this principle.

Bevis-Gibson Flash Light Torsion Dynamometer

Fig. 8.34 shows a Bevis-Gibson torsion dynamometer which consists of two discs *A* and *B* fixed on a shaft at points as far apart as possible. Each disc has a narrow radial slot and the two slots are in line when there is no torque transmitted along the shaft. A powerful electric lamp *L* is fixed to the bearing cap of the shaft behind disc *A*. The lamp is masked so as to throw a narrow pencil of light parallel to the axis of shaft. Also this lamp has a slot directly opposite to the slot of disc *A*. Behind the disc *B*, an eye-piece *E* is fitted to the shaft bearing. This eye-piece is capable of slight circumferential adjustment.

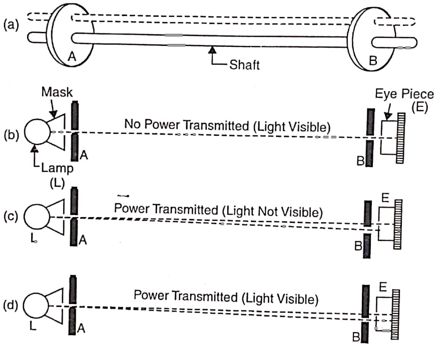


Fig. 8.34

The eye-piece is adjusted so as to receive the ray of light which passes from the lamp through the slots in the two discs, when the shaft is at rest. When the shaft rotates without transmitting any torque, a ray of light will be received in the eye-piece once per revolution as shown in Fig. 8.34 (*b*). But when shaft is rotating and torque is transmitted, the shaft twists and the slot in the disc *B* sights its position. Due to this, the ray of light does not reach to the eye-piece as shown in Fig. 8.34 (*c*). If now the eye-piece is displaced along the circular are by an amount equal to the lag of disc *B* by means of vernier, then the ray of light will be visible in eye- piece as shown in Fig. 8.34 (*d*). Hence the angular displacement of the eye-piece and therefore the angle of twist of the shaft may be measured upto $\frac{1}{100}$th of a degree.

16

Turning Moment Diagrams and Flywheel

Features

16.1. Introduction

The turning moment diagram (also known as ***crank-effort diagram***) is the graphical representation of the turning moment or crank-effort for various positions of the crank. It is plotted on cartesian co-ordinates, in which the turning moment is taken as the ordinate and crank angle as abscissa.

16.2. Turning Moment Diagram for a Single Cylinder Double Acting Steam Engine

A turning moment diagram for a single cylinder double acting steam engine is shown in Fig. 16.1. The vertical ordinate represents the turning moment and the horizontal ordinate represents the crank angle.

We have discussed in Chapter 15 (Art. 15.10.) that the turning moment on the crankshaft,

$$T = F_P \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$$

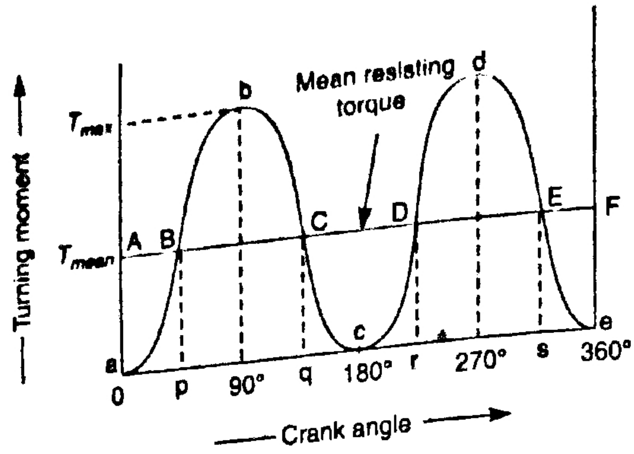


Fig. 16.1. Turning moment diagram for a single cylinder, double acting steam engine.

where F_P = Piston effort,

r = Radius of crank,

n = Ratio of the connecting rod length and radius of crank, and

θ = Angle turned by the crank from inner dead centre.

From the above expression, we see that the turning moment (T) is zero, when the crank angle (θ) is zero. It is maximum when the crank angle is 90° and it is again zero when crank angle is 180°.

This is shown by the curve *abc* in Fig. 16.1 and it represents the turning moment diagram for outstroke. The curve *cde* is the turning moment diagram for instroke and is somewhat similar to the curve *abc*.

Since the work done is the product of the turning moment and the angle turned, therefore the area of the turning moment diagram represents the work done per revolution. In actual practice, the engine is assumed to work against the mean resisting torque, as shown by a horizontal line *AF*. The height of the ordinate *a A* represents the mean height of the turning moment diagram. Since it is assumed that the work done by the turning moment per revolution is equal to the work done against the mean resisting torque, therefore the area of the rectangle *aAFe* is proportional to the work done against the mean resisting torque.

For flywheel, have a look at your tailor's manual sewing machine.

Notes: 1. When the turning moment is positive (*i.e.* when the engine torque is more than the mean resisting torque) as shown between points *B* and *C* (or *D* and *E*) in Fig. 16.1, the crankshaft accelerates and the work is done by the steam.

2. When the turning moment is negative (*i.e.* when the engine torque is less than the mean resisting torque) as shown between points C and D in Fig. 16.1, the crankshaft retards and the work is done on the system.

3. If T = Torque on the crankshaft at any instant, and

T_{mean} = Mean resisting torque.

Then accelerating torque on the rotating parts of the engine

$$= T - T_{mean}$$

4. If $(T - T_{mean})$ is positive, the flywheel accelerates and if $(T - T_{mean})$ is negative, then the flywheel retards.

16.3. Turning Moment Diagram for a Four Stroke Cycle Internal Combustion Engine

A turning moment diagram for a four stroke cycle internal combustion engine is shown in Fig. 16.2. We know that in a four stroke cycle internal combustion engine, there is one working stroke after the crank has turned through two revolutions, *i.e.* 720° (or 4 π radians).

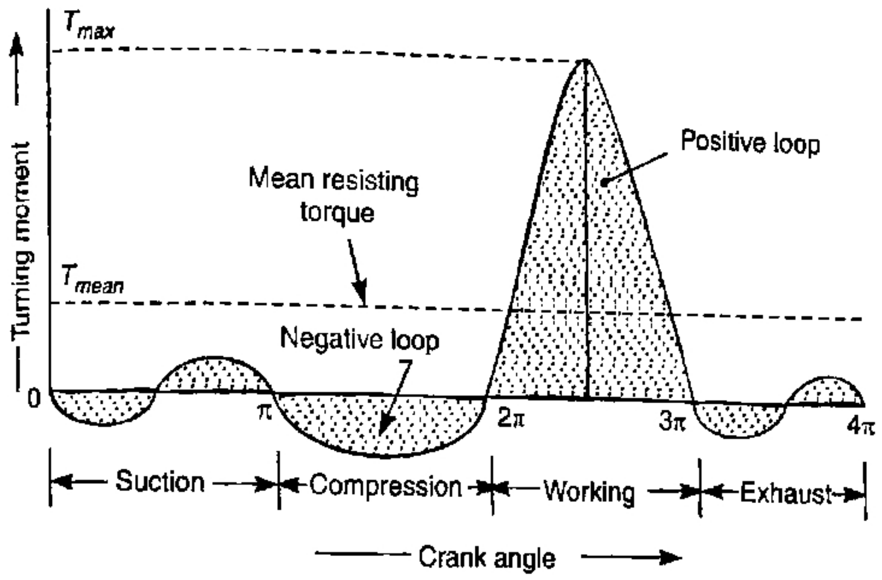


Fig. 16.2. Turning moment diagram for a four stroke cycle internal combustion engine.

Since the pressure inside the engine cylinder is less than the atmospheric pressure during the suction stroke, therefore a negative loop is formed as shown in Fig. 16.2. During the compression stroke, the work is done on the gases, therefore a higher negative loop is obtained. During the expansion or working stroke, the fuel burns and the gases expand, therefore a large positive loop is obtained. In this stroke, the work is done by the gases. During exhaust stroke, the work is done on the gases, therefore a negative loop is formed. It may be noted that the effect of the inertia forces on the piston is taken into account in Fig. 16.2.

16.4. Turning Moment Diagram for a Multi-cylinder Engine

A separate turning moment diagram for a compound steam engine having three cylinders and the resultant turning moment diagram is shown in Fig. 16.3. The resultant turning moment diagram is the sum of the turning moment diagrams for the three cylinders. It may be noted that the first cylinder is the high pressure cylinder, second cylinder is the intermediate cylinder and the third cylinder is the low pressure cylinder. The cranks, in case of three cylinders, are usually placed at 120° to each other.

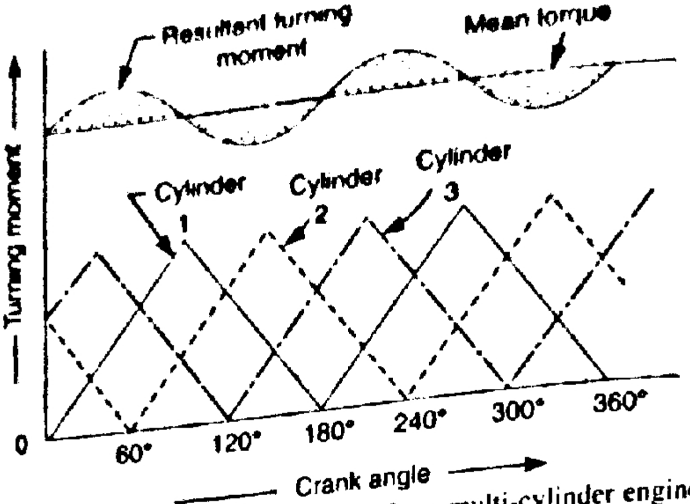


Fig. 16.3. Turning moment diagram for a multi-cylinder engine.

16.5. Fluctuation of Energy

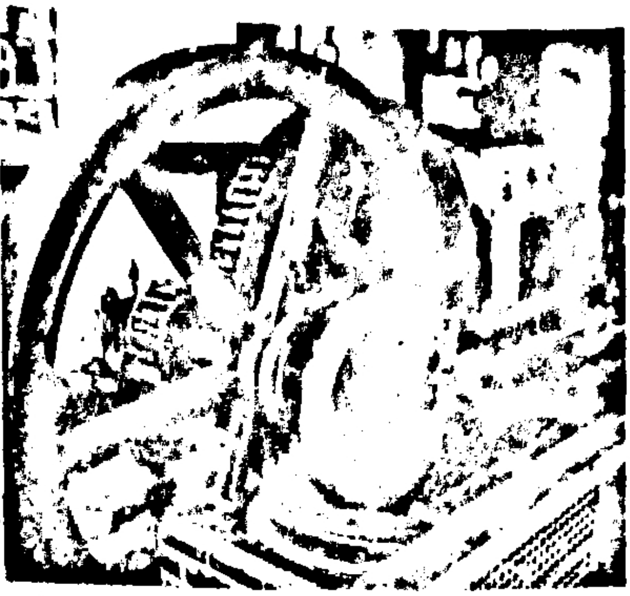
The fluctuation of energy may be determined by the turning moment diagram for one complete cycle of operation. Consider the turning moment diagram for a single cylinder double acting steam engine as shown in Fig. 16.1. We see that the mean resisting torque line *AF* cuts the turning moment diagram at points *B, C, D* and *E*. When the crank moves from *a* to *p*, the work done by the engine is equal to the area *aBp*, whereas the energy required is represented by the area *aABp*. In other words, the engine has done less work (equal to the area *a AB*) than the requirement. This amount of energy is taken from the flywheel and hence the speed of the flywheel decreases. Now the crank moves from *p* to *q*, the work done by the engine is equal to the area *pBbCq*, whereas the requirement of energy is represented by the area *pBCq*. Therefore, the engine has done more work than the requirement. This excess work (equal to the area *BbC*) is stored in the flywheel and hence the speed of the flywheel increases while the crank moves from *p* to *q*.

Similarly, when the crank moves from *q* to *r*, more work is taken from the engine than is developed. This loss of work is represented by the area *C c D*. To supply this loss, the flywheel gives up some of its energy and thus the speed decreases while the crank moves from *q* to *r*. As the crank moves from *r* to *s*, excess energy is again developed given by the area *D d E* and the speed again increases. As the piston moves from *s* to *e*, again there is a loss of work and the speed decreases. The variations of energy above and below the mean resisting torque line are called ***fluctuations of energy***. The areas *BbC, CcD, DdE*, etc. represent fluctuations of energy.

A little consideration will show that the engine has a maximum speed either at *q* or at *s*. This is due to the fact that the flywheel absorbs energy while the crank moves from *p* to *q* and from *r* to *s*. On the other hand, the engine has a minimum speed either at *p* or at *r*. The reason is that the flywheel gives out some of its energy when the crank moves from *a* to *p* and *q* to *r*. The difference between the maximum and the minimum energies is known as ***maximum fluctuation of energy***.

16.6. Determination of Maximum Fluctuation of Energy

A turning moment diagram for a multi-cylinder engine is shown by a wavy curve in Fig. 16.4. The horizontal line *AG* represents the mean torque line. Let a_1, a_3, a_5 be the areas above the mean torque line and a_2, a_4 and a_6 be the areas below the mean torque line. These areas represent some quantity of energy which is either added or subtracted from the energy of the moving parts of the engine.



A flywheel stores energy when the supply is in excess and releases energy when energy is in deficit.

Let the energy in the flywheel at $A = E$, then from Fig. 16.4, we have

Energy at $B = E + a_1$

Energy at $C = E + a_1 - a_2$

Energy at $D = E + a_1 - a_2 + a_3$

Energy at $E = E + a_1 - a_2 + a_3 - a_4$

Energy at $F = E + a_1 - a_2 + a_3 - a_4 + a_5$

Energy at $G = E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

$= $ Energy at A (*i.e.* cycle repeats after G)

Let us now suppose that the greatest of these energies is at B and least at E. Therefore,

Maximum energy in flywheel

$$= E + a_1$$

Minimum energy in the flywheel

$$= E + a_1 - a_2 + a_3 - a_4$$

∴ Maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + a_1) - (E + a_1 - a_2 + a_3 - a_4) = a_2 - a_3 + a_4$$

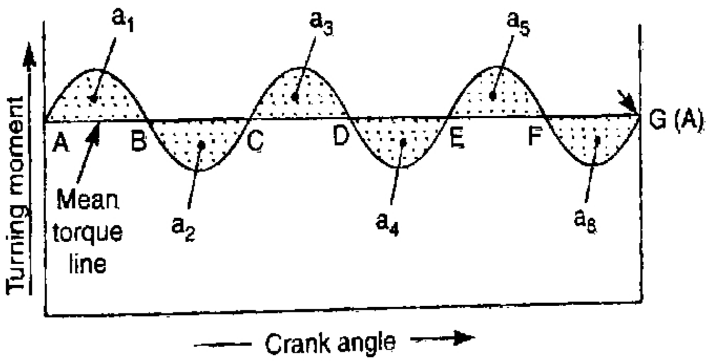


Fig. 16.4. Determination of maximum fluctuation of energy.

16.7. Coefficient of Fluctuation of Energy

It may be defined as the **ratio of the maximum fluctuation of energy to the work done per cycle.** Mathematically, coefficient of fluctuation of energy,

$$C_E = \frac{\text{Maximum fluctuation of energy}}{\text{Work done per cycle}}$$

The work done per cycle (in N-m or joules) may be obtained by using the following two relations :

1. Work done per cycle $= T_{mean} \times \theta$

where
T_{mean} = Mean torque, and

θ = Angle turned (in radians), in one revolution.

$= 2\pi$, in case of steam engine and two stroke internal combustion engines

$= 4\pi$, in case of four stroke internal combustion engines.

The mean torque (T_{mean}) in N-m may be obtained by using the following relation :

$$T_{mean} = \frac{P \times 60}{2\pi N} = \frac{P}{\omega}$$

where P = Power transmitted in watts,

N = Speed in r.p.m., and

ω = Angular speed in rad/s $= 2\pi N/60$

2. The work done per cycle may also be obtained by using the following relation :

$$\text{Work done per cycle} = \frac{P \times 60}{n}$$

where n = Number of working strokes per minute,

= N, in case of steam engines and two stroke internal combustion engines,

= $N/2$, in case of four stroke internal combustion engines.

The following table shows the values of coefficient of fluctuation of energy for steam engines and internal combustion engines.

Table 16.1. Coefficient of fluctuation of energy (C_E) for steam and internal combustion engines.

S.No.	*Type of engine*	*Coefficient of fluctuation of energy (C_E)*
1.	Single cylinder, double acting steam engine	0.21
2.	Cross-compound steam engine	0.096
3.	Single cylinder, single acting, four stroke gas engine	1.93
4.	Four cylinders, single acting, four stroke gas engine	0.066
5.	Six cylinders, single acting, four stroke gas engine	0.031

16.8. Flywheel

A flywheel used in machines serves as a reservoir, which stores energy during the period when the supply of energy is more than the requirement, and releases it during the period when the requirement of energy is more than the supply.

In case of steam engines, internal combustion engines, reciprocating compressors and pumps, the energy is developed during one stroke and the engine is to run for the whole cycle on the energy produced during this one stroke. For example, in internal combustion engines, the energy is developed only during expansion or power stroke which is much more than the engine load and no energy is being developed during suction, compression and exhaust strokes in case of four stroke engines and during compression in case of two stroke engines. The excess energy developed during power stroke is absorbed by the flywheel and releases it to the crankshaft during other strokes in which no energy is developed, thus rotating the crankshaft at a uniform speed. A little consideration will show that when the flywheel absorbs energy, its speed increases and when it releases energy, the speed decreases. Hence a flywheel does not maintain a constant speed, it simply reduces the fluctuation of speed. In other words, ***a flywheel controls the speed variations caused by the fluctuation of the engine turning moment during each cycle of operation.***

In machines where the operation is intermittent like *punching machines, shearing machines, rivetting machines, crushers, etc., the flywheel stores energy from the power source during the greater portion of the operating cycle and gives it up during a small period of the cycle. Thus, the energy from the power source to the machines is supplied practically at a constant rate throughout the operation.

Note: The function of a **governor in an engine is entirely different from that of a flywheel. It regulates the mean speed of an engine when there are variations in the load, *e.g.*, when the load on the engine increases, it becomes necessary to increase the supply of working fluid. On the other hand, when the load decreases, less working fluid is required. The governor automatically controls the supply of working fluid to the engine with the varying load condition and keeps the mean speed of the engine within certain limits.

As discussed above, the flywheel does not maintain a constant speed, it simply reduces the fluctuation of speed. It does not control the speed variations caused by the varying load.

16.9. Coefficient of Fluctuation of Speed

The difference between the maximum and minimum speeds during a cycle is called the *maximum fluctuation of speed*. The ratio of the maximum fluctuation of speed to the mean speed is called the ***coefficient of fluctuation of speed.***

Let N_1 and N_2 = Maximum and minimum speeds in r.p.m. during the cycle, and

$$N = \text{Mean speed in r.p.m.} = \frac{N_1 + N_2}{2}$$

∴ Coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = \frac{2(N_1 - N_2)}{N_1 + N_2}$$

$$= \frac{\omega_1 - \omega_2}{\omega} = \frac{2(\omega_1 - \omega_2)}{\omega_1 + \omega_2} \quad \text{...(In terms of angular speeds)}$$

$$= \frac{v_1 - v_2}{v} = \frac{2(v_1 - v_2)}{v_1 + v_2} \quad \text{...(In terms of linear speeds)}$$

The coefficient of fluctuation of speed is a limiting factor in the design of flywheel. It varies depending upon the nature of service to which the flywheel is employed.

Note. The reciprocal of the coefficient of fluctuation of speed is known as ***coefficient of steadiness*** and is denoted by m.

$$\therefore \quad m = \frac{1}{C_S} = \frac{N}{N_1 - N_2}$$

16.10. Energy Stored in a Flywheel

A flywheel is shown in Fig. 16.5. We have discussed in Art. 16.5 that when a flywheel absorbs energy, its speed increases and when it gives up energy, its speed decreases.

Let m = Mass of the flywheel in kg,

k = Radius of gyration of the flywheel in metres,

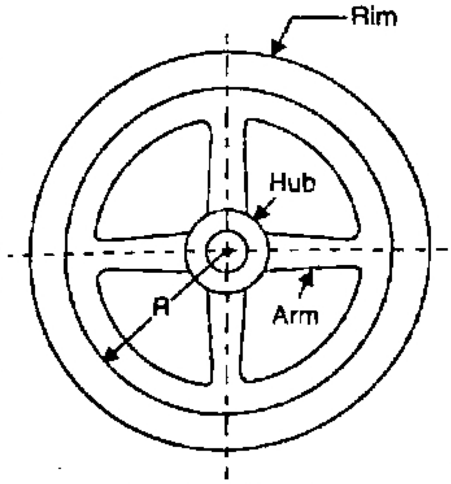


Fig. 16.5. Flywheel.

* See Art. 16.12.

** See Chapter 18 on Governors.

I = Mass moment of inertia of the flywheel about its axis of rotation in kg m² = $m.k^2$.

N_1 and N_2 = Maximum and minimum speeds during the cycle in r.p.m.

ω_1 and ω_2 = Maximum and minimum angular speeds during the cycle in rad/s,

N = Mean speed during the cycle in r.p.m. $= \frac{N_1 + N_2}{2}$,

ω = Mean angular speed during the cycle in rad/s $= \frac{\omega_1 + \omega_2}{2}$,

C_S = Coefficient of fluctuation of speed, $= \frac{N_1 - N_2}{N}$ or $\frac{\omega_1 - \omega_2}{\omega}$

We know that the mean kinetic energy of the flywheel,

$$E = \frac{1}{2} \times I.\omega^2 = \frac{1}{2} \times m.k^2.\omega^2 \quad \text{(in N-m or joules)}$$

As the speed of the flywheel changes from ω_1 to ω_2, the maximum fluctuation of energy,

$$\Delta E = \text{Maximum K.E.} - \text{Minimum K.E.}$$

$$= \frac{1}{2} \times I(\omega_1)^2 - \frac{1}{2} \times I(\omega_2)^2 = \frac{1}{2} \times I\left[(\omega_1)^2 - (\omega_2)^2\right]$$

$$= \frac{1}{2} \times I(\omega_1 + \omega_2)(\omega_1 - \omega_2) = I.\omega(\omega_1 - \omega_2) \quad \text{...(i)}$$

$$\text{...}\left(\because \omega = \frac{\omega_1 + \omega_2}{2}\right)$$

$$= I.\omega^2\left(\frac{\omega_1 - \omega_2}{\omega}\right) \quad \text{... (Multiplying and dividing by } \omega)$$

$$= I.\omega^2.C_S = m.k^2.\omega^2.C_S \quad \text{... } (\because I = m.k^2) \quad \text{...(ii)}$$

$$= 2.E.C_S \text{ (in N-m or joules)} \quad \text{...}\left(\because E = \frac{1}{2} \times I.\omega^2\right) \text{...(iii)}$$

The radius of gyration (k) may be taken equal to the mean radius of the rim (R), because the thickness of rim is very small as compared to the diameter of rim. Therefore, substituting $k = R$, in equation (*ii*), we have

$$\Delta E = m.R^2.\omega^2.C_S = m.v^2.C_S$$

where v = Mean linear velocity (*i.e.* at the mean radius) in m/s = $\omega.R$

Notes. 1. Since $\omega = 2\pi N/60$, therefore equation (*i*) may be written as

$$\Delta E = I \times \frac{2\pi N}{60}\left(\frac{2\pi N_1}{60} - \frac{2\pi N_2}{60}\right) = \frac{4\pi^2}{3600} \times I \times N(N_1 - N_2)$$

$$= \frac{\pi^2}{900} \times m.k^2.N(N_1 - N_2)$$

$$= \frac{\pi^2}{900} \times m.k^2.N^2.C_S \quad \text{...}\left(\because C_S = \frac{N_1 - N_2}{N}\right)$$

2. In the above expressions, only the mass moment of inertia of the flywheel rim (I) is considered and the mass moment of inertia of the hub and arms is neglected. This is due to the fact that the major portion of the mass of the flywheel is in the rim and a small portion is in the hub and arms. Also the hub and arms are nearer to the axis of rotation, therefore the mass moment of inertia of the hub and arms is small.

Example 16.1. *The mass of flywheel of an engine is 6.5 tonnes and the radius of gyration is 1.8 metres. It is found from the turning moment diagram that the fluctuation of energy is 56 kN-m. If the mean speed of the engine is 120 r.p.m., find the maximum and minimum speeds.*

Solution. Given : $m = 6.5$ t $= 6500$ kg ; $k = 1.8$ m ; $\Delta E = 56$ kN-m $= 56 \times 10^3$ N-m ; $N = 120$ r.p.m.

Let N_1 and N_2 = Maximum and minimum speeds respectively.

We know that fluctuation of energy (ΔE),

$$56 \times 10^3 = \frac{\pi^2}{900} \times m.k^2 . N(N_1 - N_2) = \frac{\pi^2}{900} \times 6500\,(1.8)^2\, 120\,(N_1 - N_2)$$

$$= 27\,715\,(N_1 - N_2)$$

$$\therefore \quad N_1 - N_2 = 56 \times 10^3 / 27\,715 = 2 \text{ r.p.m.} \qquad ...(i)$$

We also know that mean speed (N),

$$120 = \frac{N_1 + N_2}{2} \text{ or } N_1 + N_2 = 120 \times 2 = 240 \text{ r.p.m.} \qquad ...(ii)$$

From equations (*i*) and (*ii*),

$N_1 = 121$ r.p.m., and $N_2 = 119$ r.p.m. **Ans.**

Example 16.2. *The flywheel of a steam engine has a radius of gyration of 1 m and mass 2500 kg. The starting torque of the steam engine is 1500 N-m and may be assumed constant. Determine:* **1.** *the angular acceleration of the flywheel, and* **2.** *the kinetic energy of the flywheel after 10 seconds from the start.*

Solution. Given : $k = 1$ m ; $m = 2500$ kg ; $T = 1500$ N-m

1. Angular acceleration of the flywheel

Let α = Angular acceleration of the flywheel.

We know that mass moment of inertia of the flywheel,

$$I = m.k^2 = 2500 \times 1^2 = 2500 \text{ kg-m}^2$$

$\therefore$ Starting torque of the engine (T),

$$1500 = I.\alpha = 2500 \times \alpha \quad \text{or} \quad \alpha = 1500 / 2500 = 0.6 \text{ rad/s}^2 \textbf{ Ans.}$$

2. Kinetic energy of the flywheel

First of all, let us find out the angular speed of the flywheel after 10 seconds from the start (i.e. from rest), assuming uniform acceleration.

Let ω_1 = Angular speed at rest = 0

ω_2 = Angular speed after 10 seconds, and

t = Time in seconds.

We know that $\omega_2 = \omega_1 + \alpha t = 0 + 0.6 \times 10 = 6$ rad/s

∴ Kinetic energy of the flywheel

$$= \frac{1}{2} \times I(\omega_2)^2 = \frac{1}{2} \times 2500 \times 6^2 = 45\,000 \text{ N-m} = 45 \text{ kN-m} \quad \textbf{Ans.}$$

Example 16.3. *A horizontal cross compound steam engine develops 300 kW at 90 r.p.m. The coefficient of fluctuation of energy as found from the turning moment diagram is to be 0.1 and the fluctuation of speed is to be kept within ± 0.5% of the mean speed. Find the weight of the flywheel required, if the radius of gyration is 2 metres.*

Solution. Given : $P = 300$ kW $= 300 \times 10^3$ W; $N = 90$ r.p.m.; $C_E = 0.1$; $k = 2$ m

We know that the mean angular speed,

$$\omega = 2\,\pi\, N/60 = 2\,\pi \times 90/60 = 9.426 \text{ rad/s}$$

Let ω_1 and ω_2 = Maximum and minimum speeds respectively.

Since the fluctuation of speed is ± 0.5% of mean speed, therefore total fluctuation of speed,

$$\omega_1 - \omega_2 = 1\%\ \omega = 0.01\ \omega$$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 0.01$$

We know that work done per cycle

$$= P \times 60 / N = 300 \times 10^3 \times 60 / 90 = 200 \times 10^3 \text{ N-m}$$

∴ Maximum fluctuation of energy,

$$\Delta E = \text{Work done per cycle} \times C_E = 200 \times 10^3 \times 0.1 = 20 \times 10^3 \text{ N-m}$$

Let m = Mass of the flywheel.

We know that maximum fluctuation of energy (ΔE),

$$20 \times 10^3 = m.k^2.\omega^2.C_S = m \times 2^2 \times (9.426)^2 \times 0.01 = 3.554\, m$$

$$\therefore \quad m = 20 \times 10^3/3.554 = 5630 \text{ kg} \quad \textbf{Ans.}$$

Example 16.4. *The turning moment diagram for a petrol engine is drawn to the following scales : Turning moment, 1 mm = 5 N-m ; crank angle, 1 mm = 1°. The turning moment diagram repeats itself at every half revolution of the engine and the areas above and below the mean turning moment line taken in order are 295, 685, 40, 340, 960, 270 mm². The rotating parts are equivalent to a mass of 36 kg at a radius of gyration of 150 mm. Determine the coefficient of fluctuation of speed when the engine runs at 1800 r.p.m.*

Solution. Given : $m = 36$ kg ; $k = 150$ mm $= 0.15$ m ; $N = 1800$ r.p.m. or $\omega = 2\,\pi \times 1800/60 = 188.52$ rad /s

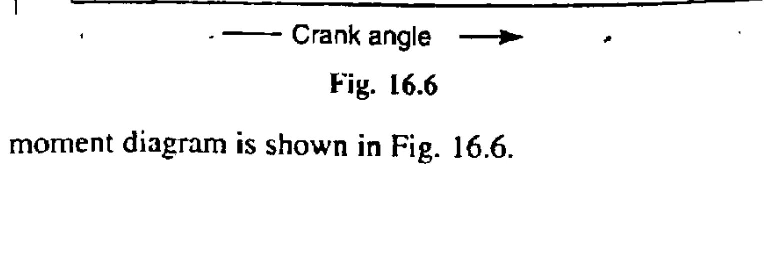


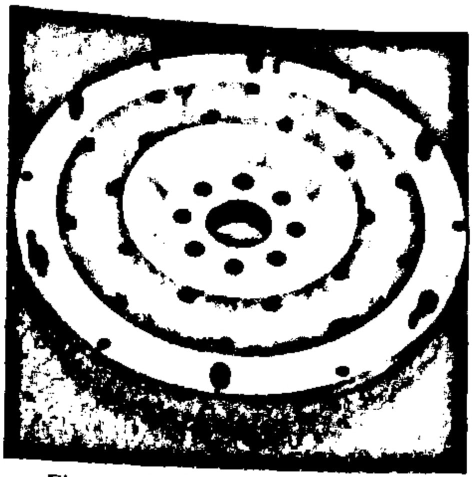
Fig. 16.6

The turning moment diagram is shown in Fig. 16.6.

Since the turning moment scale is 1 mm = 5 N-m and angle scale is 1 mm = 1° = π /180 rad, therefore,

1 mm² on turning moment diagram

$$= 5 \times \frac{\pi}{180} = \frac{\pi}{36} \text{ N-m}$$



Flywheel of an electric motor.

Let the total energy at $A = E$, then referring to Fig. 16.6.

$$\text{Energy at } B = E + 295 \quad \text{... (Maximum energy)}$$

$$\text{Energy at } C = E + 295 - 685 = E - 390$$

$$\text{Energy at } D = E - 390 + 40 = E - 350$$

$$\text{Energy at } E = E - 350 - 340 = E - 690 \quad \text{...(Minimum energy)}$$

$$\text{Energy at } F = E - 690 + 960 = E + 270$$

$$\text{Energy at } G = E + 270 - 270 = E = \text{Energy at } A$$

We know that maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + 295) - (E - 690) = 985 \text{ mm}^2$$

$$= 985 \times \frac{\pi}{36} = 86 \text{ N-m} = 86 \text{ J}$$

Let C_S = Coefficient of fluctuation of speed.

We know that maximum fluctuation of energy (ΔE),

$$86 = m.k^2\omega^2.C_S = 36 \times (0.15)^2 \times (188.52)^2\, C_S = 28\,787\, C_S$$

$$\therefore \quad C_S = 86 / 28\,787 = 0.003 \text{ or } 0.3\% \quad \textbf{Ans.}$$

Example 16.5. *The turning moment diagram for a multicylinder engine has been drawn to scale 1 mm = 600 N-m vertically and 1 mm = 3° horizontally. The intercepted areas between the output torque curve and the mean resistance line, taken in order from one end, are as follows :*

+ 52, – 124, + 92, – 140, + 85, – 72 and + 107 mm², when the engine is running at a speed of 600 r.p.m. If the total fluctuation of speed is not to exceed ± 1.5% of the mean, find the necessary mass of the flywheel of radius 0.5 m.

Solution. Given : $N = 600$ r.p.m. or $\omega = 2\pi \times 600 / 60 = 62.84$ rad / s ; $R = 0.5$ m

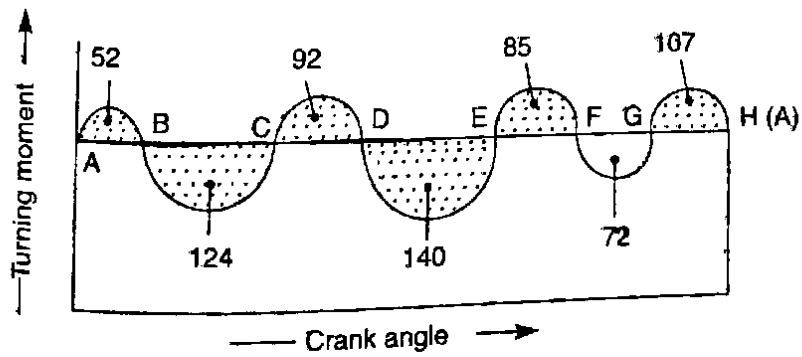


Fig. 16.7

Since the total fluctuation of speed is not to exceed ± 1.5% of the mean speed, therefore

$$\omega_1 - \omega_2 = 3\%\ \omega = 0.03\ \omega$$

and coefficient of fluctuation of speed,

$$C_s = \frac{\omega_1 - \omega_2}{\omega} = 0.03$$

The turning moment diagram is shown in Fig. 16.7.

Since the turning moment scale is 1 mm = 600 N-m and crank angle scale is 1 mm = 3° = 3° × π/180 = π / 60 rad, therefore

1 mm² on turning moment diagram

$$= 600 \times \pi/60 = 31.42 \text{ N-m}$$

Let the total energy at $A = E$, then referring to Fig. 16.7,

Energy at $B = E + 52$ ...(Maximum energy)

Energy at $C = E + 52 - 124 = E - 72$

Energy at $D = E - 72 + 92 = E + 20$

Energy at $E = E + 20 - 140 = E - 120$ ...(Minimum energy)

Energy at $F = E - 120 + 85 = E - 35$

Energy at $G = E - 35 - 72 = E - 107$

Energy at $H = E - 107 + 107 = E$ = Energy at A

We know that maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$
$$= (E + 52) - (E - 120) = 172 = 172 \times 31.42 = 5404 \text{ N-m}$$

Let m = Mass of the flywheel in kg.

We know that maximum fluctuation of energy (ΔE),

$$5404 = m.R^2.\omega^2.C_S = m \times (0.5)^2 \times (62.84)^2 \times 0.03 = 29.6\, m$$

$$\therefore \quad m = 5404 / 29.6 = 183 \text{ kg} \quad \textbf{Ans.}$$

Example 16.6. *A shaft fitted with a flywheel rotates at 250 r.p.m. and drives a machine. The torque of machine varies in a cyclic manner over a period of 3 revolutions. The torque rises from 750 N-m to 3000 N-m uniformly during 1/2 revolution and remains constant for the following revolution. It then falls uniformly to 750 N-m during the next 1/2 revolution and remains constant for one revolution, the cycle being repeated thereafter.*

Determine the power required to drive the machine and percentage fluctuation in speed, if the driving torque applied to the shaft is constant and the mass of the flywheel is 500 kg with radius of gyration of 600 mm.

Solution. Given : $N = 250$ r.p.m. or $\omega = 2\pi \times 250/60 = 26.2$ rad/s ; $m = 500$ kg ; $k = 600$ mm $= 0.6$ m

The turning moment diagram for the complete cycle is shown in Fig. 16.8.

We know that the torque required for one complete cycle

$$= \text{Area of figure } OABCDEF$$
$$= \text{Area } OAEF + \text{Area } ABG + \text{Area } BCHG + \text{Area } CDH$$
$$= OF \times OA + \frac{1}{2} \times AG \times BG + GH \times CH + \frac{1}{2} \times HD \times CH$$

$$= 6\pi \times 750 + \frac{1}{2} \times \pi(3000 - 750) + 2\pi(3000 - 750) + \frac{1}{2} \times \pi(3000 - 750)$$

$$= 11\,250\,\pi \text{ N-m} \quad ...(i)$$

If T_{mean} is the mean torque in N-m, then torque required for one complete cycle

$$= T_{mean} \times 6\pi \text{ N-m} \quad ...(ii)$$

From equations (*i*) and (*ii*),

$$T_{mean} = 11\,250\,\pi / 6\,\pi = 1875 \text{ N-m}$$

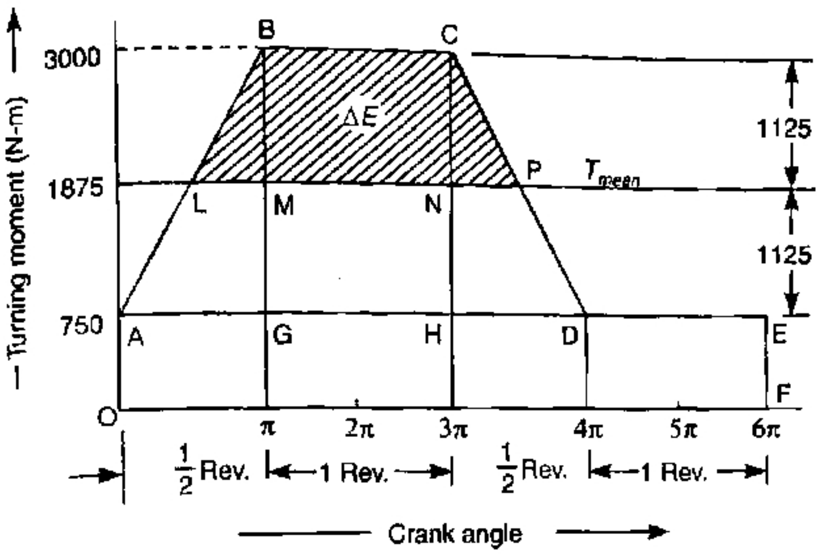


Fig. 16.8

Power required to drive the machine

We know that power required to drive the machine,

$$P = T_{mean} \times \omega = 1875 \times 26.2 = 49\,125 \text{ W} = 49.125 \text{ kW} \quad \textbf{Ans.}$$

Coefficient of fluctuation of speed

Let C_S = Coefficient of fluctuation of speed.

First of all, let us find the values of LM and NP. From similar triangles ABG and BLM,

$$\frac{LM}{AG} = \frac{BM}{BG} \quad \text{or} \quad \frac{LM}{\pi} = \frac{3000 - 1875}{3000 - 750} = 0.5 \quad \text{or} \quad LM = 0.5\,\pi$$

Now, from similar triangles CHD and CNP,

$$\frac{NP}{HD} = \frac{CN}{CH} \quad \text{or} \quad \frac{NP}{\pi} = \frac{3000 - 1875}{3000 - 750} = 0.5 \quad \text{or} \quad NP = 0.5\,\pi$$

From Fig. 16.8, we find that

$$BM = CN = 3000 - 1875 = 1125 \text{ N-m}$$

Since the area above the mean torque line represents the maximum fluctuation of energy, therefore, maximum fluctuation of energy,

$$\Delta E = \text{Area } LBCP = \text{Area } LBM + \text{Area } MBCN + \text{Area } PNC$$

$$= \frac{1}{2} \times LM \times BM + MN \times BM + \frac{1}{2} \times NP \times CN$$

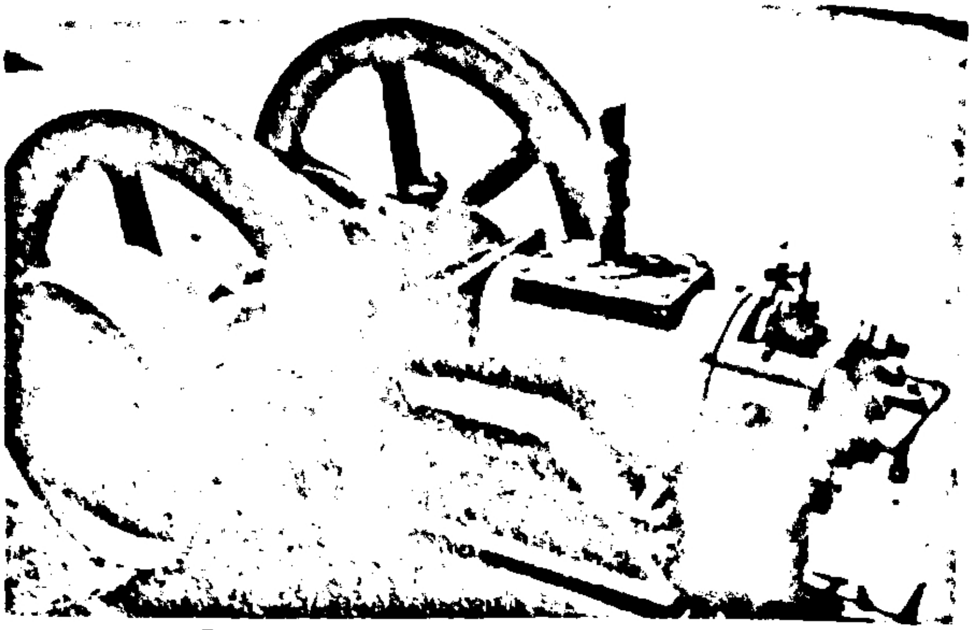
$$= \frac{1}{2} \times 0.5\pi \times 1125 + 2\pi \times 1125 + \frac{1}{2} \times 0.5\pi \times 1125$$

$$= 8837 \text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$8837 = m.k^2.\omega^2.C_S = 500 \times (0.6)^2 \times (26.2)^2 \times C_S = 123\,559\, C_S$$

$$C_S = \frac{8837}{123559} = 0.071 \textbf{ Ans.}$$



Flywheel of a pump run by a diesel engine.

Example 16.7. *During forward stroke of the piston of the double acting steam engine, the turning moment has the maximum value of 2000 N-m when the crank makes an angle of 80° with the inner dead centre. During the backward stroke, the maximum turning moment is 1500 N-m when the crank makes an angle of 80° with the outer dead centre. The turning moment diagram for the engine may be assumed for simplicity to be represented by two triangles.*

If the crank makes 100 r.p.m. and the radius of gyration of the flywheel is 1.75 m, find the coefficient of fluctuation of energy and the mass of the flywheel to keep the speed within ± 0.75% of the mean speed. Also determine the crank angle at which the speed has its minimum and maximum values.

Solution. Given : $N = 100$ r.p.m. or $\omega = 2\pi \times 100/60 = 10.47$ rad /s; $k = 1.75$ m

Since the fluctuation of speed is ± 0.75% of mean speed, therefore total fluctuation of speed,

$$\omega_1 - \omega_2 = 1.5\%\ \omega$$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 1.5\% = 0.015$$

Coefficient of fluctuation of energy

The turning moment diagram for the engine during forward and backward strokes is shown in Fig. 16.9. The point O represents the inner dead centre (I.D.C.) and point G represents the outer dead centre (O.D.C). We know that maximum turning moment when crank makes an angle of 80° (or $80 \times \pi / 180 = 4\pi/9$ rad) with I.D.C.,

$$\therefore \quad AB = 2000 \text{ N-m}$$

maximum turning moment when crank makes an angle of 80° with outer dead centre (O.D.C.) or $180° + 80° = 260° = 260 \times \pi/180 = 13\pi/9$ rad with I.D.C.,

$$LM = 1500 \text{ N-m}$$

Let

$$T_{mean} = EB = QM = \text{Mean resisting torque.}$$

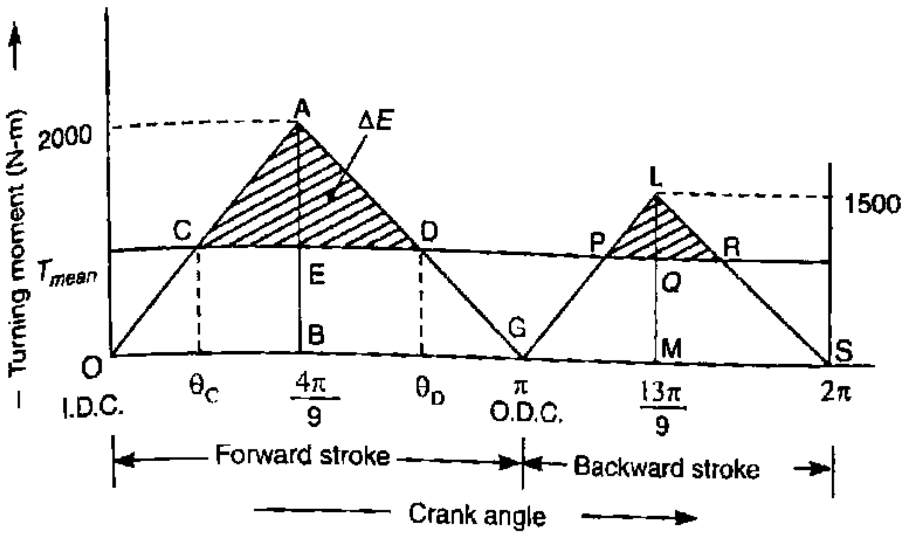


Fig. 16.9

We know that work done per cycle

$$= \text{Area of triangle } OAG + \text{Area of triangle } GLS$$

$$= \frac{1}{2} \times OG \times AB + \frac{1}{2} \times GS \times LM$$

$$= \frac{1}{2} \times \pi \times 2000 + \frac{1}{2} \times \pi \times 1500 = 1750\pi \text{ N-m} \quad \text{...}(i)$$

We also know that work done per cycle

$$= T_{mean} \times 2\pi \text{ N-m} \quad \text{...}(ii)$$

From equations (*i*) and (*ii*),

$$T_{mean} = 1750\pi / 2\pi = 875 \text{ N-m}$$

From similar triangles ACD and AOG,

$$\frac{CD}{AE} = \frac{OG}{AB}$$

or

$$CD = \frac{OG}{AB} \times AE = \frac{OG}{AB}(AB - EB) = \frac{\pi}{2000}(2000 - 875) = 1.764 \text{ rad}$$

∴ Maximum fluctuation of energy,

$$\Delta E = \text{Area of triangle } ACD = \frac{1}{2} \times CD \times AE$$

$$= \frac{1}{2} \times CD(AB - EB) = \frac{1}{2} \times 1.764(2000 - 875) = 992 \text{ N-m}$$

We know that coefficient of fluctuation of energy,

$$C_E = \frac{\text{Max.fluctuation of energy}}{\text{Work done per cycle}} = \frac{992}{1750\pi} = 0.18 \text{ or } 18\% \quad \textbf{Ans.}$$

Mass of the flywheel

Let m = Mass of the flywheel.

We know that maximum fluctuation of energy (ΔE),

$$992 = m.k^2.\omega^2.C_S = m \times (1.75)^2 \times (10.47)^2 \times 0.015 = 5.03\ m$$

$$\therefore \quad m = 992 / 5.03 = 197.2 \text{ kg} \quad \textbf{Ans.}$$

Crank angles for the minimum and maximum speeds

We know that the speed of the flywheel is minimum at point C and maximum at point D (See Art. 16.5).

Let θ_C and θ_D = Crank angles from I.D.C., for the minimum and maximum speeds.

Flywheel of small steam engine.

From similar triangles ACE and AOB,

$$\frac{CE}{OB} = \frac{AE}{AB}$$

or

$$CE = \frac{AE}{AB} \times OB = \frac{AB - EB}{AB} \times OB = \frac{2000 - 875}{2000} \times \frac{4\pi}{9} = \frac{\pi}{4} \text{ rad}$$

$$\therefore \quad \theta_C = \frac{4\pi}{9} - \frac{\pi}{4} = \frac{7\pi}{36} \text{ rad} = \frac{7\pi}{36} \times \frac{180}{\pi} = 35° \quad \textbf{Ans.}$$

Again from similar triangles AED and ABG,

$$\frac{ED}{BG} = \frac{AE}{AB}$$

or

$$ED = \frac{AE}{AB} \times BG = \frac{AB - EB}{AB}(OG - OB)$$

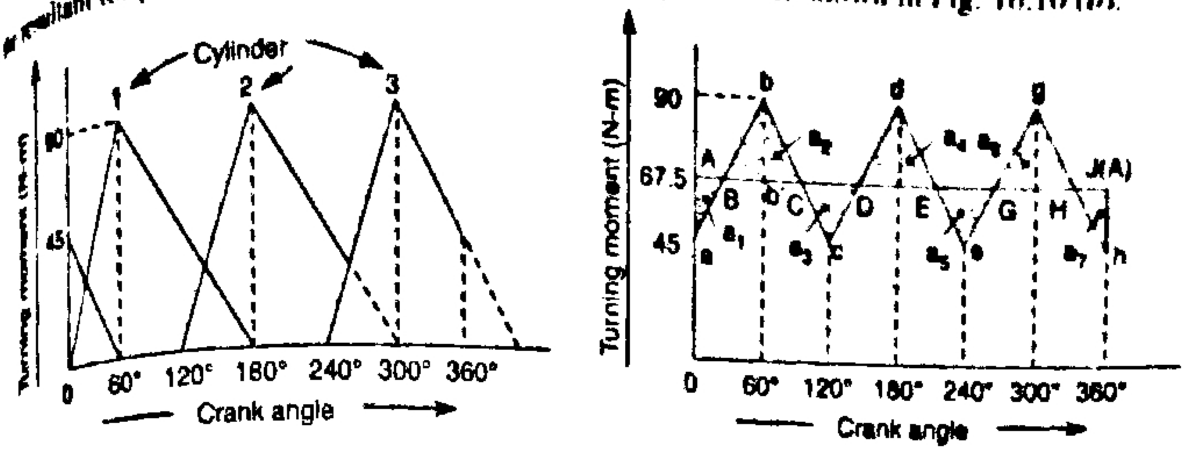
$$= \frac{2000 - 875}{2000}\left(\pi - \frac{4\pi}{9}\right) = \frac{2.8\pi}{9} \text{ rad}$$

$$\therefore \quad \theta_D = \frac{4\pi}{9} + \frac{2.8\pi}{9} = \frac{6.8\pi}{9} \text{ rad} = \frac{6.8\pi}{9} \times \frac{180}{\pi} = 136° \textbf{ Ans.}$$

Example 16.8. *A three cylinder single acting engine has its cranks set equally at 120° and it runs at 600 r.p.m. The torque-crank angle diagram for each cycle is a triangle for the power stroke with a maximum torque of 90 N-m at 60° from dead centre of corresponding crank. The torque on the return stroke is sensibly zero. Determine : 1. power developed, 2. coefficient of fluctuation of speed if the mass of the flywheel is 12 kg and has a radius of gyration of 80 mm, 3. coefficient of fluctuation of energy, and 4. maximum angular acceleration of the flywheel.*

Solution. Given : N = 600 r.p.m. or $\omega = 2\pi \times 600/60 = 62.84$ rad/s; T_{max} = 90 N-m; m = 12 kg; k = 80 mm = 0.08 m

The torque-crank angle diagram for the individual cylinders is shown in Fig 16.10 (*a*), and resultant torque-crank angle diagram for the three cylinders is shown in Fig. 16.10 (*b*).



(*a*) (*b*)

Fig. 16.10

1. ***Power developed***

We know that work done/cycle

$$= \text{Area of three triangles} = 3 \times \frac{1}{2} \times \pi \times 90 = 424 \text{ N-m}$$

and mean torque,
$$T_{mean} = \frac{\text{Work done / cycle}}{\text{Crank angle / cycle}} = \frac{424}{2\pi} = 67.5 \text{ N-m}$$

$\therefore$ Power developed $= T_{mean} \times \omega = 67.5 \times 62.84 = 4240 \text{ W} = 4.24 \text{ kW}$ **Ans.**

2. ***Coefficient of fluctuation of speed***

Let C_S = Coefficient of fluctuation of speed.

First of all, let us find the maximum fluctuation of energy (ΔE).

From Fig. 16.10 (*b*), we find that

$$a_1 = \text{Area of triangle } AaB = \frac{1}{2} \times AB \times Aa$$

$$= \frac{1}{2} \times \frac{\pi}{6} \times (67.5 - 45) = 5.89 \text{ N-m} = a_7 \quad \ldots(\because AB = 30° = \pi/6 \text{ rad})$$

$$a_2 = \text{Area of triangle } BbC = \frac{1}{2} \times BC \times bb'$$

$$= \frac{1}{2} \times \frac{\pi}{3}(90 - 67.5) = 11.78 \text{ N-m} \quad \ldots(\because BC = 60° = \pi/3 \text{ rad})$$

$$= a_3 = a_4 = a_5 = a_6$$

Now, let the total energy at $A = E$, then referring to Fig. 16.10 (*b*),

Energy at $B = E - 5.89$

Energy at $C = E - 5.89 + 11.78 = E + 5.89$

Energy at $D = E + 5.89 - 11.78 = E - 5.89$

Energy at $E = E - 5.89 + 11.78 = E + 5.89$

Energy at $G = E + 5.89 - 11.78 = E - 5.89$

Energy at $H = E - 5.89 + 11.78 = E + 5.89$

Energy at $J = E + 5.89 - 5.89 = E$ = Energy at A

From above we see that maximum energy

$$= E + 5.89$$

and minimum energy $= E - 5.89$

∴ * Maximum fluctuation of energy,

$$\Delta E = (E + 5.89) - (E - 5.89) = 11.78 \text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$11.78 = m.k^2.\omega^2.C_S = 12 \times (0.08)^2 \times (62.84)^2 \times C_S = 303.3\, C_S$$

$$\therefore \quad C_S = 11.78 / 303.3 = 0.04 \text{ or } 4\% \quad \textbf{Ans.}$$

3. ***Coefficient of fluctuation of energy***

We know that coefficient of fluctuation of energy,

$$C_E = \frac{\text{Max. fluctuation of energy}}{\text{Work done/cycle}} = \frac{11.78}{424} = 0.0278 = 2.78\% \quad \textbf{Ans.}$$

4. ***Maximum angular acceleration of the flywheel***

Let α = Maximum angular acceleration of the flywheel.

We know that,

$$T_{max} - T_{mean} = I.\alpha = m.k^2.\alpha$$

$$90 - 67.5 = 12 \times (0.08)^2 \times \alpha = 0.077\, \alpha$$

$$\therefore \quad \alpha = \frac{90 - 67.5}{0.077} = 292 \text{ rad/s}^2 \quad \textbf{Ans.}$$

Example 16.9. *A single cylinder, single acting, four stroke gas engine develops 20 kW at 300 r.p.m. The work done by the gases during the expansion stroke is three times the work done on the gases during the compression stroke, the work done during the suction and exhaust strokes being negligible. If the total fluctuation of speed is not to exceed ± 2 per cent of the mean speed and the turning moment diagram during compression and expansion is assumed to be triangular in shape, find the moment of inertia of the flywheel.*

Solution. Given : $P = 20$ kW $= 20 \times 10^3$ W; $N = 300$ r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s

Since the total fluctuation of speed ($\omega_1 - \omega_2$) is not to exceed ± 2 per cent of the mean speed (ω), therefore

$$\omega_1 - \omega_2 = 4\%\ \omega$$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 4\% = 0.04$$

The turning moment-crank angle diagram for a four stroke engine is shown in Fig. 16.11. It is assumed to be triangular during compression and expansion strokes, neglecting the suction and exhaust strokes.

---

* Since the area above the mean torque line represents the maximum fluctuation of energy, therefore maximum fluctuation of energy,

$$\Delta E = \text{Area } Bbc = \text{Area } DdE = \text{Area } Ggh$$

$$= \frac{1}{2} \times \frac{\pi}{3} (90 - 67.5) = 11.78 \text{ N-m}$$

We know that for a four stroke engine, number of working strokes per cycle,

$$n = N/2 = 300/2 = 150$$

∴ Work done/cycle $= P \times 60/n = 20 \times 10^3 \times 60/150 = 8000$ N-m ...(*i*)

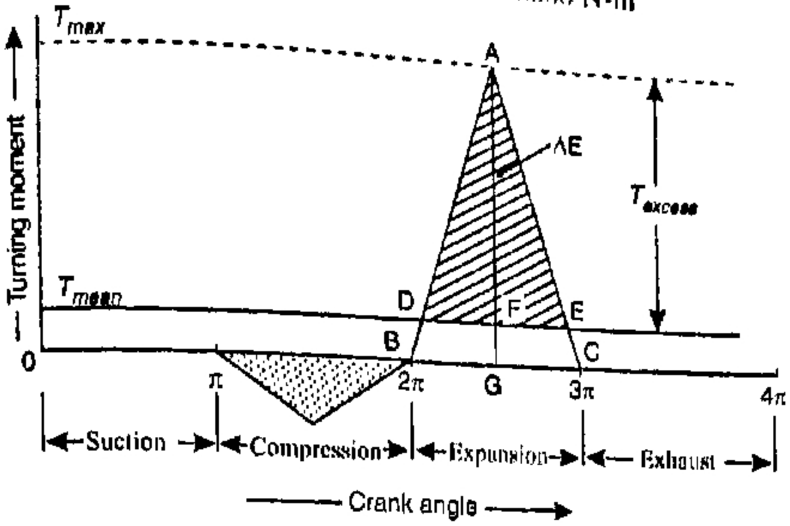


Fig. 16.11

Since the work done during suction and exhaust strokes is negligible, therefore net work done per cycle (during compression and expansion strokes)

$$= W_E - W_C = W_E - \frac{W_E}{3} = \frac{2}{3} W_E \qquad \text{... } (\because W_E = 3W_C) \text{ ...(ii)}$$

Equating equations (*i*) and (*ii*), work done during expansion stroke,

$$W_E = 8000 \times 3/2 = 12\,000 \text{ N-m}$$

We know that work done during expansion stroke (W_E),

$$12\,000 = \text{Area of triangle } ABC = \frac{1}{2} \times BC \times AG = \frac{1}{2} \times \pi \times AG$$

$$\therefore \quad AG = T_{max} = 12\,000 \times 2/\pi = 7638 \text{ N-m}$$

and mean turning moment,

$$^*T_{mean} = FG = \frac{\text{Work done/cycle}}{\text{Crank angle/cycle}} = \frac{8000}{4\pi} = 637 \text{ N-m}$$

∴ Excess turning moment,

$$T_{excess} = AF = AG - FG = 7638 - 637 = 7001 \text{ N-m}$$

Now, from similar triangles *ADE* and *ABC*,

$$\frac{DE}{BC} = \frac{AF}{AG} \quad \text{or} \quad DE = \frac{AF}{AG} \times BC = \frac{7001}{7638} \times \pi = 2.88 \text{ rad}$$

Since the area above the mean turning moment line represents the maximum fluctuation of energy, therefore maximum fluctuation of energy,

$$\Delta E = \text{Area of } \Delta\, ADE = \frac{1}{2} \times DE \times AF = \frac{1}{2} \times 2.88 \times 7001 = 10081 \text{ N-m}$$

* The mean turning moment (T_{mean}) may also be obtained by using the following relation :

$$P = T_{mean} \times \omega \text{ or } T_{mean} = P/\omega = 20 \times 10^3/31.42 = 637 \text{ N-m}$$

Let I = Moment of inertia of the flywheel in kg-m².

We know that maximum fluctuation of energy (ΔE),

$$10\,081 = I.\omega^2.C_s = I \times (31.42)^2 \times 0.04 = 39.5\,I$$

$$\therefore \quad I = 10081/39.5 = 255.2 \text{ kg-m}^2 \quad \textbf{Ans.}$$

Example 16.10. *The turning moment diagram for a four stroke gas engine may be assumed for simplicity to be represented by four triangles, the areas of which from the line of zero pressure are as follows :*

Suction stroke = 0.45×10^{-3} m²; Compression stroke = 1.7×10^{-3} m²; Expansion stroke = 6.8×10^{-3} m²; Exhaust stroke = 0.65×10^{-3} m². Each m² of area represents 3 MN-m of energy.

Assuming the resisting torque to be uniform, find the mass of the rim of a flywheel required to keep the speed between 202 and 198 r.p.m. The mean radius of the rim is 1.2 m.

Solution. Given : $a_1 = 0.45 \times 10^{-3}$ m² ; $a_2 = 1.7 \times 10^{-3}$ m² ; $a_3 = 6.8 \times 10^{-3}$ m²; $a_4 = 0.65 \times 10^{-3}$ m²; $N_1 = 202$ r.p.m; $N_2 = 198$ r.p.m.; $R = 1.2$ m

The turning moment-crank angle diagram for a four stroke engine is shown in Fig. 16.12. The areas below the zero line of pressure are taken as negative while the areas above the zero line of pressure are taken as positive.

$$\therefore \quad \text{Net area} = a_3 - (a_1 + a_2 + a_4)$$

$$= 6.8 \times 10^{-3} - (0.45 \times 10^{-3} + 1.7 \times 10^{-3} + 0.65 \times 10^{-3}) = 4 \times 10^{-3} \text{ m}^2$$

Since the energy scale is 1 m² = 3 MN-m = 3×10^6 N-m, therefore,

Net work done per cycle = $4 \times 10^{-3} \times 3 \times 10^6 = 12 \times 10^3$ N-m ...(*i*)

We also know that work done per cycle,

$$= T_{mean} \times 4\pi \text{ N-m} \quad ...(ii)$$

From equations (*i*) and (*ii*),

$$T_{mean} = FG = 12 \times 10^3/4\pi = 955 \text{ N-m}$$

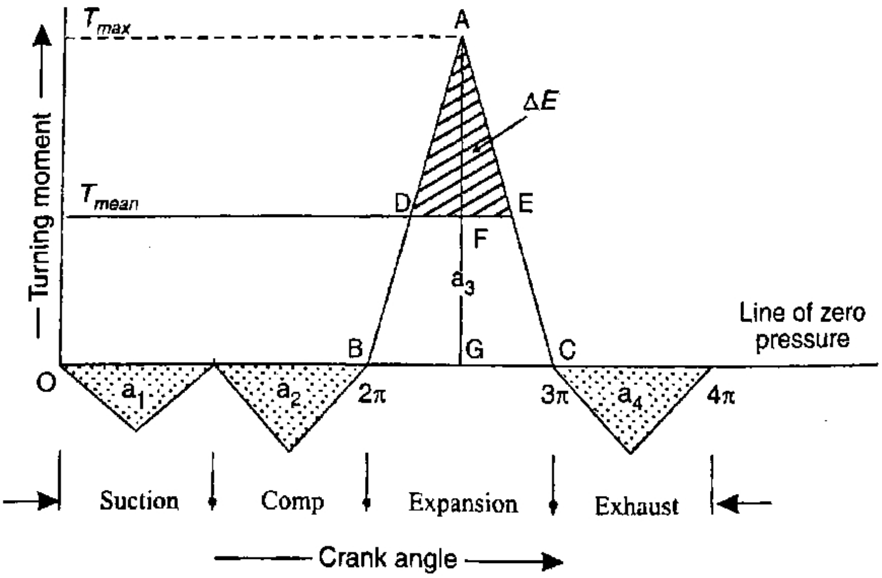


Fig. 16.12

Work done during expansion stroke

$$= a_3 \times \text{Energy scale} = 6.8 \times 10^{-3} \times 3 \times 10^6 = 20.4 \times 10^3 \text{ N-m} \quad ...(iii)$$

Also, work done during expansion stroke

$$= \text{Area of triangle } ABC$$

$$= \frac{1}{2} \times BC \times AG = \frac{1}{2} \times \pi \times AG = 1.571 \times AG \qquad \ldots (iv)$$

From equations (*iii*) and (*iv*),

$$AG = 20.4 \times 10^3/1.571 = 12\,985 \text{ N-m}$$

∴ Excess torque,

$$T_{excess} = AF = AG - FG = 12\,985 - 955 = 12\,030 \text{ N-m}$$

Now from similar triangles *ADE* and *ABC*,

$$\frac{DE}{BC} = \frac{AF}{AG} \quad \text{or} \quad DE = \frac{AF}{AG} \times BC = \frac{12\,030}{12\,985} \times \pi = 2.9 \text{ rad}$$

We know that the maximum fluctuation of energy,

$$\Delta E = \text{Area of } \Delta ADE = \frac{1}{2} \times DE \times AF = \frac{1}{2} \times 2.9 \times 12030 \text{ N-m}$$

$$= 17\,444 \text{ N-m}$$

Mass of the rim of a flywheel

Let m = Mass of the rim of a flywheel in kg, and

N = Mean speed of the flywheel

$$= \frac{N_1 + N_2}{2} = \frac{202 + 198}{2} = 200 \text{ r.p.m.}$$

We know that the maximum fluctuation of energy (ΔE),

$$17\,444 = \frac{\pi^2}{900} \times m.R^2.N\,(N_1 - N_2) = \frac{\pi^2}{900} \times m\,(1.2)^2\, 200 \times (202 - 198)$$

$$= 12{,}63\ m$$

∴ $m = 17\,444\,/12.36 = 1381$ kg **Ans.**

Example 16.11. *The turning moment curve for an engine is represented by the equation, T = (20 000 + 9500 sin 2θ – 5700 cos 2θ) N-m, where θ is the angle moved by the crank from inner dead centre. If the resisting torque is constant, find:*

1. Power developed by the engine ; 2. Moment of inertia of flywheel in kg-m², if the total fluctuation of speed is not to exceed 1% of mean speed which is 180 r.p.m; and 3. Angular acceleration of the flywheel when the crank has turned through 45° from inner dead centre.

Solution. Given : $T = (20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta)$ N-m ; $N = 180$ r.p.m. or $\omega = 2\pi \times 180/60 = 18.85$ rad/s

Since the total fluctuation of speed $(\omega_1 - \omega_2)$ is 1% of mean speed (ω), therefore coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 1\% = 0.01$$

Power developed by the engine

We know that work done per revolution

$$= \int_0^{2\pi} T\,d\theta = \int_0^{2\pi} (20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta)\,d\theta$$

$$= \left[20\,000\theta - \frac{9500\cos 2\theta}{2} - \frac{5700\sin 2\theta}{2} \right]_0^{2\pi}$$

$$= 20\,000 \times 2\pi = 40\,000\,\pi \text{ N-m}$$

and mean resisting torque of the engine,

$$T_{mean} = \frac{\text{Work done per revolution}}{2\pi} = \frac{40\,000\pi}{2\pi} = 20000 \text{ N-m}$$

We know that power developed by the engine

$$= T_{mean} \cdot \omega = 20\,000 \times 18.85 = 377\,000 \text{ W} = 377 \text{ kW} \quad \textbf{Ans.}$$

2. ***Moment of inertia of the flywheel***

Let I = Moment of inertia of the flywheel in kg-m^2.

The turning moment diagram for one stroke (*i.e.* half revolution of the crankshaft) is shown in Fig. 16.13. Since at points B and D, the torque exerted on the crankshaft is equal to the mean resisting torque on the flywheel, therefore,

$$T = T_{mean}$$

$$20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta = 20\,000$$

or $9500 \sin 2\theta = 5700 \cos 2\theta$

$$\tan 2\theta = \sin 2\theta / \cos 2\theta = 5700/9500 = 0.6$$

$\therefore$ $2\theta = 31°$ or $\theta = 15.5°$

$\therefore$ $\theta_B = 15.5°$ and $\theta_D = 90° + 15.5° = 105.5°$

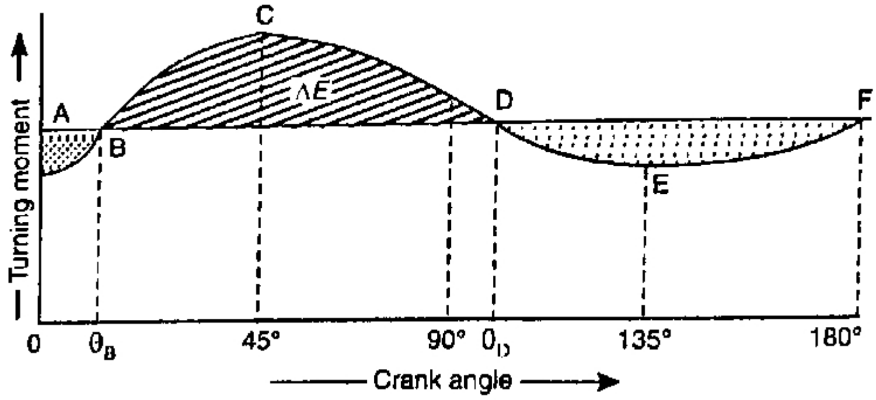


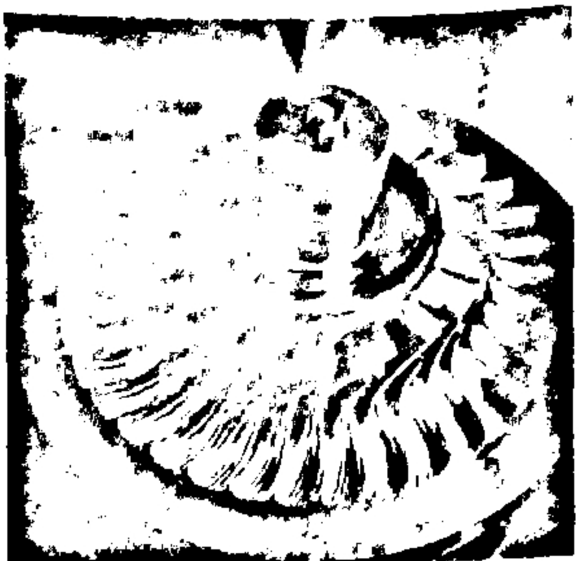
Fig. 16.13

Maximum fluctuation of energy,

$$\Delta E = \int_{\theta_B}^{\theta_D} \left(T - T_{mean} \right) d\theta$$

$$= \int_{15.5°}^{105.5°} (20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta - 20\,000)\, d\theta$$

$$= \left[-\frac{9500\cos 2\theta}{2} - \frac{5700\sin 2\theta}{2} \right]_{15.5°}^{105.5°} = 11\,078 \text{ N-m}$$



Nowadays steam turbines like this can be produced entirely by computer-controlled machine tools, directly from the engineer's computer.

Note : This picture is given as additional information.

We know that maximum fluctuation of energy (ΔE),

$$11\,078 = I.\omega^2.C_S = I \times (18.85)^2 \times 0.01 = 3.55\, I$$

$\therefore$ $I = 11078/3.55 = 3121$ kg-m^2 **Ans.**

3. Angular acceleration of the flywheel

Let α = Angular acceleration of the flywheel, and

θ = Angle turned by the crank from inner dead centre = 45° ... (Given)

The angular acceleration in the flywheel is produced by the excess torque over the mean torque. We know that excess torque at any instant,

$$T_{excess} = T - T_{mean}$$
$$= 20000 + 9500 \sin 2\theta - 5700 \cos 2\theta - 20000$$
$$= 9500 \sin 2\theta - 5700 \cos 2\theta$$

$\therefore$ Excess torque at 45°

$$= 9500 \sin 90° - 5700 \cos 90° = 9500 \text{ N-m} \quad \ldots (i)$$

We also know that excess torque

$$= I.\alpha = 3121 \times \alpha \quad \ldots (ii)$$

From equations (*i*) and (*ii*),

$$\alpha = 9500/3121 = 3.044 \text{ rad/s}^2 \textbf{ Ans.}$$

Example 16.12. *A certain machine requires a torque of (5000 + 500 sin θ) N-m to drive it, where θ is the angle of rotation of shaft measured from certain datum. The machine is directly coupled to an engine which produces a torque of (5000 + 600 sin 2θ) N-m. The flywheel and the other rotating parts attached to the engine has a mass of 500 kg at a radius of gyration of 0.4 m. If the mean speed is 150 r.p.m., find :* **1.** *the fluctuation of energy,* **2.** *the total percentage fluctuation of speed, and* **3.** *the maximum and minimum angular acceleration of the flywheel and the corresponding shaft position.*

Solution. Given : $T_1 = (5000 + 500 \sin \theta)$ N-m ; $T_2 = (5000 + 600 \sin 2\theta)$ N-m ; $m = 500$ kg; $k = 0.4$ m ; $N = 150$ r.p.m. or $\omega = 2\pi \times 150/60 = 15.71$ rad/s

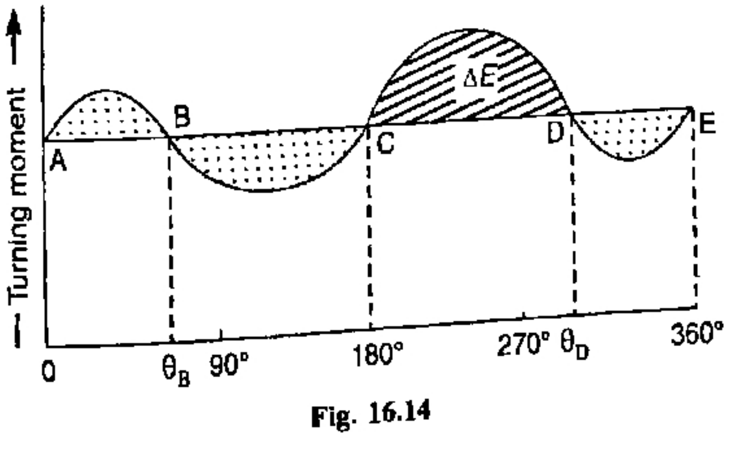


Fig. 16.14

1. ***Fluctuation of energy***

We know that change in torque

$$= T_2 - T_1 = (5000 + 600 \sin 2\theta) - (5000 + 500 \sin \theta)$$
$$= 600 \sin 2\theta - 500 \sin \theta$$

This change is zero when

$$600 \sin 2\theta = 500 \sin \theta \quad \text{or} \quad 1.2 \sin 2\theta = \sin \theta$$

$$1.2 \times 2 \sin \theta \cos \theta = \sin \theta \quad \text{or} \quad 2.4 \sin \theta \cos \theta = \sin \theta \quad \ldots (\because \sin 2\theta = 2 \sin \theta \cos \theta)$$

∴ Either $\sin \theta = 0$ or $\cos \theta = 1/2.4 = 0.4167$

when $\sin \theta = 0,\ \theta = 0°,\ 180°$ and $360°$

i.e. $\theta_A = 0°,\ \theta_C = 180°$ and $\theta_E = 360°$

when $\cos \theta = 0.4167,\ \theta = 65.4°$ and $294.6°$

i.e. $\theta_B = 65.4°$ and $\theta_D = 294.6°$

The turning moment diagram is shown in Fig. 16.14. The maximum fluctuation of energy lies between C and D (*i.e.* between 180° and 294.6°), as shown shaded in Fig. 16.14.

∴ Maximum fluctuation of energy,

$$\Delta E = \int_{180°}^{294.6°} \left(T_2 - T_1\right) d\theta$$

$$= \int_{180°}^{294.6°} \left[(5000 + 600 \sin 2\theta) - (5000 + 500 \sin \theta)\right] d\theta$$

$$= \left[-\frac{600 \cos 2\theta}{2} + 500 \cos \theta\right]_{180°}^{294.6°} = 1204 \text{ N-m} \textbf{ Ans.}$$

2. ***Total percentage fluctuation of speed***

Let C_S = Total percentage fluctuation of speed.

We know that maximum fluctuation of energy (ΔE),

$$1204 = m.k^2.\omega^2.C_S = 500 \times (0.4)^2 \times (15.71)^2 \times C_S = 19\,744\ C_S$$

$$\therefore \quad C_S = 1204 / 19\,744 = 0.061 \quad \text{or} \quad 6.1\% \textbf{ Ans.}$$

3. ***Maximum and minimum angular acceleration of the flywheel and the corresponding shaft positions***

The change in torque must be maximum or minimum when acceleration is maximum or minimum. We know that

Change in torque, $T = T_2 - T_1 = (5000 + 600 \sin 2\theta) - (5000 + 500 \sin \theta)$

$$= 600 \sin 2\theta - 500 \sin \theta \quad \ldots (i)$$

Differentiating this expression with respect to θ and equating to zero for maximum or minimum values.

$$\therefore \quad \frac{d}{d\theta}(600 \sin 2\theta - 500 \sin \theta) = 0 \quad \text{or} \quad 1200 \cos 2\theta - 500 \cos \theta = 0$$

or $12 \cos 2\theta - 5 \cos \theta = 0$

$$12(2\cos^2\theta - 1) - 5\cos\theta = 0 \qquad \ldots (\because \cos 2\theta = 2\cos^2\theta - 1)$$

$$24\cos^2\theta - 5\cos\theta - 12 = 0$$

$$\therefore \quad \cos\theta = \frac{5 \pm \sqrt{25 + 4 \times 12 \times 24}}{2 \times 24} = \frac{5 \pm 34.3}{48}$$

$$= 0.8187 \quad \text{or} \quad -0.6104$$

$$\therefore \quad \theta = 35° \quad \text{or} \quad 127.6° \quad \textbf{Ans.}$$

Substituting $\theta = 35°$ in equation (*i*), we have maximum torque,

$$T_{max} = 600 \sin 70° - 500 \sin 35° = 277 \text{ N-m}$$

Substituting $\theta = 127.6°$ in equation (*i*), we have minimum torque,

$$T_{min} = 600 \sin 255.2° - 500 \sin 127.6° = -976 \text{ N-m}$$

We know that maximum acceleration,

$$\alpha_{max} = \frac{T_{max}}{I} = \frac{277}{500 \times (0.4)^2} = 3.46 \text{ rad/s}^2 \quad \textbf{Ans.} \qquad \ldots (\because I = m.k^2)$$

and minimum acceleration (or maximum retardation),

$$\alpha_{min} = \frac{T_{min}}{I} = \frac{976}{500 \times (0.4)^2} = 12.2 \text{ rad/s}^2 \quad \textbf{Ans.}$$

Example 16.13. *The equation of the turning moment curve of a three crank engine is (5000 + 1500 sin 3 θ) N-m, where θ is the crank angle in radians. The moment of inertia of the flywheel is 1000 kg-m² and the mean speed is 300 r.p.m. Calculate : 1. power of the engine, and 2. the maximum fluctuation of the speed of the flywheel in percentage when (i) the resisting torque is constant, and (ii) the resisting torque is (5000 + 600 sin θ) N-m.*

Solution. Given : $T = (5000 + 1500 \sin 3\theta)$ N-m ; $I = 1000$ kg-m² ; $N = 300$ r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s

1. *Power of the engine*

We know that work done per revolution

$$= \int_0^{2\pi} (5000 + 1500 \sin 3\theta)\, d\theta = \left[5000\theta - \frac{1500 \cos 3\theta}{3}\right]_0^{2\pi}$$

$$= 10\,000\pi \text{ N-m}$$

$\therefore$ Mean resisting torque,

$$T_{mean} = \frac{\text{Work done/rev}}{2\pi} = \frac{10\,000\pi}{2\pi} = 5000 \text{ N-m}$$

We know that power of the engine,

$$P = T_{mean} \cdot \omega = 5000 \times 31.42 = 157\,100 \text{ W} = 157.1 \text{ kW} \quad \textbf{Ans.}$$

2. *Maximum fluctuation of the speed of the flywheel*

Let

$$C_S = \text{Maximum or total fluctuation of speed of the flywheel.}$$

(i) When resisting torque is constant

The turning moment diagram is shown in Fig. 16.15. Since the resisting torque is constant therefore the torque exerted on the shaft is equal to the mean resisting torque on the flywheel

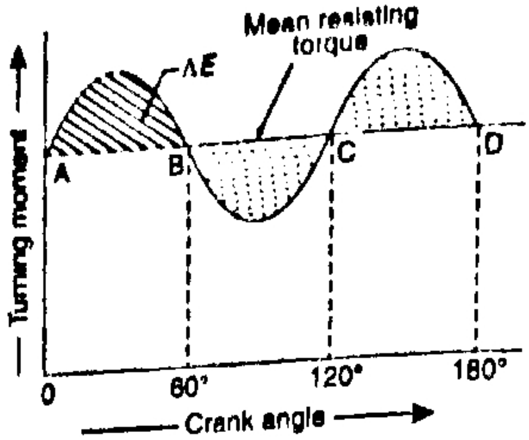


Fig. 16.15

$$\therefore \quad T = T_{mean}$$

$$5000 + 1500 \sin 3\theta = 5000$$

$$1500 \sin 3\theta = 0 \quad \text{or} \quad \sin 3\theta = 0$$

$$\therefore \quad 3\theta = 0° \quad \text{or} \quad 180°$$

$$\theta = 0° \quad \text{or} \quad 60°$$

∴ Maximum fluctuation of energy,

$$\Delta E = \int_0^{60°} (T - T_{mean})\, d\theta = \int_0^{60°} (5000 + 1500 \sin 3\theta - 5000)\, d\theta$$

$$= \int_0^{60°} 1500 \sin 3\theta\, d\theta = \left[-\frac{1500 \cos 3\theta}{3} \right]_0^{60°} = 1000 \text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$1000 = I.\omega^2.C_S = 1000 \times (31.42)^2 \times C_S = 987\,216\, C_S$$

$$\therefore \quad C_S = 1000 / 987\,216 = 0.001 \text{ or } 0.1\% \textbf{ Ans.}$$

(ii) When resisting torque is (5000 + 600 sin θ) N-m

The turning moment diagram is shown in Fig. 16.16. Since at points B and C, the torque exerted on the shaft is equal to the mean resisting torque on the flywheel, therefore

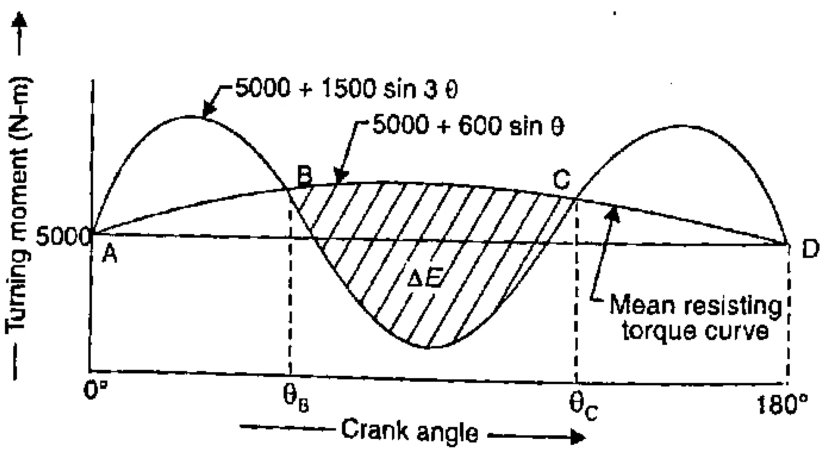


Fig. 16.16

$5000 + 1500 \sin 3\theta = 5000 + 600 \sin \theta \quad \text{or} \quad 2.5 \sin 3\theta = \sin \theta$

$2.5 (3 \sin \theta - 4 \sin^3 \theta) = \sin \theta$ ... ($\because \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$)

$3 - 4 \sin^2 \theta = 0.4$ ... (Dividing by $2.5 \sin \theta$)

$$\sin^2 \theta = \frac{3 - 0.4}{4} = 0.65 \quad \text{or} \quad \sin \theta = 0.8062$$

$$\therefore \quad \theta = 53.7° \quad \text{or} \quad 126.3° \quad \textit{i.e.} \quad \theta_B = 53.7°, \text{ and } \theta_C = 126.3°$$

$\therefore$ Maximum fluctuation of energy,

$$^*\Delta E = \int_{53.7°}^{126.3°} \left[(5000 + 1500 \sin 3\theta) - (5000 + 600 \sin \theta)\right] d\theta$$

$$= \int_{53.7°}^{126.3°} (1500 \sin 3\theta - 600 \sin \theta)\, d\theta = \left[-\frac{1500 \cos 3\theta}{3} + 600 \cos \theta \right]_{53.7°}^{126.3°}$$

$$= -1656 \text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$1656 = I.\omega^2.C_S = 1000 \times (31.42)^2 \times C_S = 987\,216\, C_S$$

$$\therefore \quad C_S = 1656 / 987\,216 = 0.00\,168 \quad \text{or} \quad 0.168\% \textbf{ Ans.}$$

16.11. Dimensions of the Flywheel Rim

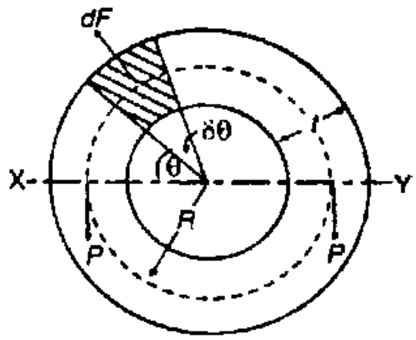


Fig. 16.17. Rim of a flywheel.

Consider a rim of the flywheel as shown in Fig. 16.17.

Let D = Mean diameter of rim in metres,

R = Mean radius of rim in metres,

A = Cross-sectional area of rim in m^2,

ρ = Density of rim material in kg/m^3,

N = Speed of the flywheel in r.p.m.,

ω = Angular velocity of the flywheel in rad/s,

v = Linear velocity at the mean radius in m/s $= \omega.R = \pi D.N/60$, and

σ = Tensile stress or hoop stress in N/m^2 due to the centrifugal force.

Consider a small element of the rim as shown shaded in Fig. 16.17. Let it subtends an angle $\delta\theta$ at the centre of the flywheel.

Volume of the small element

$$= A \times R.\delta\theta$$

$\therefore$ Mass of the small element

$$dm = \text{Density} \times \text{volume} = \rho.A.R.\delta\theta$$

and centrifugal force on the element, acting radially outwards,

$$dF = dm.\omega^2.R = \rho.A.R^2.\omega^2.\delta\theta$$

---

* Since the fluctuation of energy is negative, therefore it is shown below the mean resisting torque curve, in Fig. 16.16.

Vertical component of dF

$$= dF.\sin\theta = \rho.A.R^2.\omega^2.\delta\theta.\sin\theta$$

∴ Total vertical upward force tending to burst the rim across the diameter XY.

$$= \rho.A.R^2.\omega^2 \int_0^{\pi} \sin\theta.d\theta = \rho.A.R^2.\omega^2 \left[-\cos\theta\right]_0^{\pi}$$

$$= 2\rho.A.R^2.\omega^2 \quad \ldots(i)$$

This vertical upward force will produce tensile stress or hoop stress (also called centrifugal stress or circumferential stress), and it is resisted by $2P$, such that

$$2P = 2\,\sigma.A \quad \ldots(ii)$$

Equating equations (*i*) and (*ii*),

$$2.\rho.A.R^2.\omega^2 = 2\sigma.A$$

or $$\sigma = \rho.R^2.\omega^2 = \rho.v^2 \quad \ldots(\because v = \omega.R)$$

∴ $$v = \sqrt{\frac{\sigma}{\rho}} \quad \ldots(iii)$$

We know that mass of the rim,

$$m = \text{Volume} \times \text{density} = \pi\, D.A.\rho$$

∴ $$A = \frac{m}{\pi.D.\rho} \quad \ldots(iv)$$

From equations (*iii*) and (*iv*), we may find the value of the mean radius and cross-sectional area of the rim.

Note: If the cross-section of the rim is a rectangular, then

$$A = b \times t$$

where b = Width of the rim, and

t = Thickness of the rim.

Example 16.14. *The turning moment diagram for a multi-cylinder engine has been drawn to a scale of 1 mm to 500 N-m torque and 1 mm to 6° of crank displacement. The intercepted areas between output torque curve and mean resistance line taken in order from one end, in sq. mm are*

– 30, + 410, – 280, + 320, – 330, + 250, – 360, + 280, – 260 sq. mm, when the engine is running at 800 r.p.m.

The engine has a stroke of 300 mm and the fluctuation of speed is not to exceed ± 2% of the mean speed. Determine a suitable diameter and cross-section of the flywheel rim for a limiting value of the safe centrifugal stress of 7 MPa. The material density may be assumed as 7200 kg/m³. The width of the rim is to be 5 times the thickness.

Solution. Given : N = 800 r.p.m. or $\omega = 2\pi \times 800 / 60 = 83.8$ rad/s; *Stroke = 300 mm; $\sigma = 7$ MPa $= 7 \times 10^6$ N/m² ; $\rho = 7200$ kg/m³

Since the fluctuation of speed is ± 2% of mean speed, therefore total fluctuation of speed,

$$\omega_1 - \omega_2 = 4\%\ \omega = 0.04\ \omega$$

* Superfluous data.

...fficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 0.04$$

...er of the flywheel rim

Let $\quad D$ = Diameter of the flywheel rim in metres, and

v = Peripheral velocity of the flywheel rim in m/s.

We know that centrifugal stress (σ),

$$7 \times 10^6 = \rho.v^2 = 7200\, v^2 \quad \text{or} \quad v^2 = 7 \times 10^6/7200 = 972.2$$

$\therefore \quad v = 31.2$ m/s

We know that $\quad v = \pi\, D.N/60$

$\therefore \quad D = v \times 60 / \pi\, N = 31.2 \times 60/\pi \times 800 = 0.745$ m **Ans.**

...section of the flywheel rim

Let $\quad t$ = Thickness of the flywheel rim in metres, and

b = Width of the flywheel rim in metres = $5\,t$ ...(Given)

$\therefore$ Cross-sectional area of flywheel rim,

$$A = b.t = 5\,t \times t = 5\,t^2$$

First of all, let us find the mass (m) of the flywheel rim. The turning moment diagram is shown in Fig 16.18.

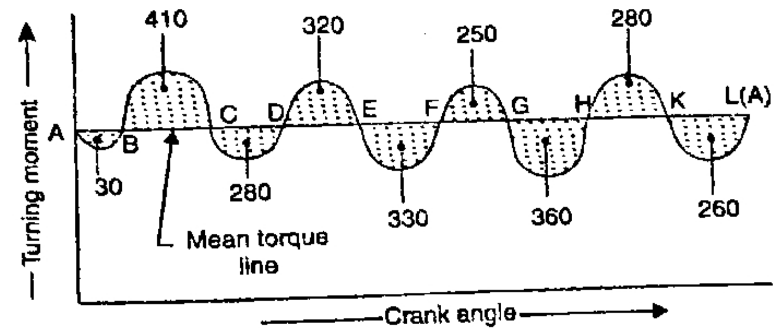


Fig. 16.18

Since the turning moment scale is 1 mm = 500 N-m and crank angle scale is 1 mm = 6° = π/30 rad, therefore

1 mm² on the turning moment diagram

$$= 500 \times \pi / 30 = 52.37 \text{ N-m}$$

Let the energy at $A = E$, then referring to Fig. 16.18,

Energy at $B = E - 30$ . . . **(Minimum energy)**

Energy at $C = E - 30 + 410 = E + 380$

Energy at $D = E + 380 - 280 = E + 100$

Energy at $E = E + 100 + 320 = E + 420$ . . . **(Maximum energy)**

Energy at $F = E + 420 - 330 = E + 90$

Energy at $G = E + 90 + 250 = E + 340$

Energy at $H = E + 340 - 360 = E - 20$

$$\text{Energy at } K = E - 20 + 280 = E + 260$$

$$\text{Energy at } L = E + 260 - 260 = E = \text{Energy at } A$$

We know that maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + 420) - (E - 30) = 450 \text{ mm}^2$$

$$= 450 \times 52.37 = 23\ 566 \text{ N-m}$$

We also know that maximum fluctuation of energy (ΔE),

$$23\ 566 = m.v^2.C_S = m \times (31.2)^2 \times 0.04 = 39\ m$$

$$\therefore \quad m = 23566 / 39 = 604 \text{ kg}$$

We know that mass of the flywheel rim (m),

$$604 = \text{Volume} \times \text{density} = \pi\ D.A.\rho$$

$$= \pi \times 0.745 \times 5t^2 \times 7200 = 84\ 268\ t^2$$

$$\therefore \quad t^2 = 604 / 84\ 268 = 0.007\ 17 \text{ m}^2 \text{ or } t = 0.085 \text{ m} = 85 \text{ mm}$$ **Ans.**

and $b = 5t = 5 \times 85 = 425$ mm **Ans.**

Example 16.15. *A single cylinder double acting steam engine develops 150 kW at a mean speed of 80 r.p.m. The coefficient of fluctuation of energy is 0.1 and the fluctuation of speed is ± 2% of mean speed. If the mean diameter of the flywheel rim is 2 metre and the hub and spokes provide 5% of the rotational inertia of the flywheel, find the mass and cross-sectional area of the flywheel rim. Assume the density of the flywheel material* (*which is cast iron*) *as 7200 kg/m³.*

Solution. Given : $P = 150$ kW $= 150 \times 10^3$ W; $N = 80$ r.p.m. or $\omega = 2\pi \times 80/60 = 8.4$ rad/s; $C_E = 0.1$; $D = 2$ m or $R = 1$ m ; $\rho = 7200$ kg/m³

Since the fluctuation of speed is ± 2% of mean speed, therefore total fluctuation of speed,

$$\omega_1 - \omega_2 = 4\%\ \omega = 0.04\ \omega$$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 0.04$$

Mass of the flywheel rim

Let m = Mass of the flywheel rim in kg, and

I = Mass moment of inertia of the flywheel in kg-m².

We know that work done per cycle

$$= P \times 60/N = 150 \times 10^3 \times 60 / 80 = 112.5 \times 10^3 \text{ N-m}$$

and maximum fluctuation of energy,

$$\Delta E = \text{Work done /cycle} \times C_E = 112.5 \times 10^3 \times 0.1 = 11\ 250 \text{ N-m}$$

We also know that maximum fluctuation of energy (ΔE),

$$11\ 250 = I.\omega^2.C_S = I \times (8.4)^2 \times 0.04 = 2.8224\ I$$

$$\therefore \quad I = 11\ 250 / 2.8224 = 3986 \text{ kg-m}^2$$

Since the hub and spokes provide 5% of the rotational inertia of the flywheel, therefore mass moment of inertia of the flywheel rim (I_{rim}) will be 95% of the flywheel, *i.e.*

$$I_{rim} = 0.95\ I = 0.95 \times 3986 = 3787 \text{ kg-m}^2$$

$I_{rim} = m.k^2$ or $\therefore m = \frac{I_{rim}}{k^2} = \frac{3787}{1^2} = 3787$ kg **Ans.** $(\because k = R)$

Cross-sectional area of the flywheel rim

Let A = Cross-sectional area of flywheel rim in m^2.

We know that the mass of the flywheel (m),

$$3787 = 2\pi R \times A \times \rho = 2\pi \times 1 \times A \times 7200 = 45\,245\,A$$

$\therefore$ $A = 3787/45\,245 = 0.084\ m^2$ **Ans.**

Example 16.16. *A multi-cylinder engine is to run at a speed of 600 r.p.m. On drawing the turning moment diagram to a scale of 1 mm = 250 N-m and 1 mm = 3°, the areas above and below the mean torque line in mm² are : + 160, – 172, + 168, – 191, + 197, – 162*

The speed is to be kept within ± 1% of the mean speed of the engine. Calculate the necessary moment of inertia of the flywheel. Determine the suitable dimensions of a rectangular flywheel rim if the breadth is twice its thickness. The density of the cast iron is 7250 kg/m³ and its hoop stress is 6 MPa. Assume that the rim contributes 92% of the flywheel effect.

Solution. Given : $N = 600$ r.p.m. or $\omega = 2\pi \times 600/60 = 62.84$ rad /s; $\rho = 7250$ kg/m³; $\sigma = 6$ MPa $= 6 \times 10^6$ N/m²

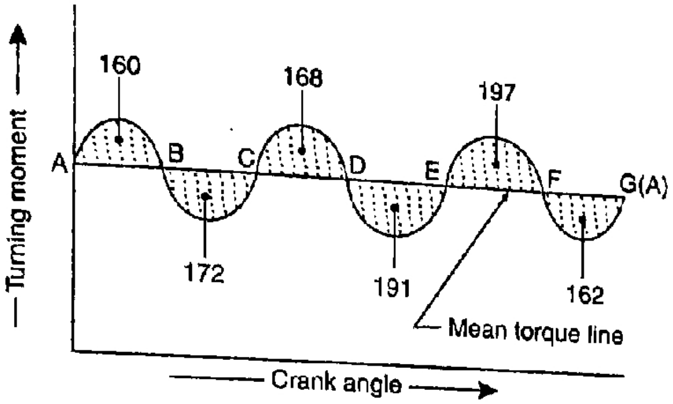


Fig. 16.19

Since the fluctuation of speed is ± 1% of mean speed, therefore, total fluctuation of speed,

$$\omega_1 - \omega_2 = 2\%\ \omega = 0.02\ \omega$$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 0.02$$

Moment of inertia of the flywheel

Let I = Moment of inertia of the flywheel in kg-m².

The turning moment diagram is shown in Fig. 16.19. The turning moment scale is 1 mm = 250 N-m and crank angle scale is 1 mm = 3° = π /60 rad, therefore,

1 mm² of turning moment diagram

$$= 250 \times \pi/60 = 13.1\ \text{N-m}$$

The mass of the flywheel rim (m) may also be obtained by using the following relation:

$\Delta E_{rim} = 0.95\ (\Delta E) = 0.95 \times 11\,250 = 10\,687.5$ N-m

$\therefore$ $\Delta E_{rim} = m.k^2.\omega^2.C_S = m\,(1)^2 \times (8.4)^2 \times 0.04 = 2.8224\,m$

$m = (\Delta E)_{rim} / 2.8224 = 10\,687.5 / 2.8224 = 3787$ kg

Let the total energy at $A = E$. Therefore from Fig. 16.19, we find that

$$\text{Energy at } B = E + 160$$

$$\text{Energy at } C = E + 160 - 172 = E - 12$$

$$\text{Energy at } D = E - 12 + 168 = E + 156$$

$$\text{Energy at } E = E + 156 - 191 = E - 35 \quad \text{... (Minimum energy)}$$

$$\text{Energy at } F = E - 35 + 197 = E + 162 \quad \text{... (Maximum energy)}$$

$$\text{Energy at } G = E + 162 - 162 = E = \text{Energy at } A$$

We know that maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + 162) - (E - 35) = 197 \text{ mm}^2$$

$$= 197 \times 13.1 = 2581 \text{ N-m}$$

We also know that maximum fluctuation of energy (ΔE),

$$2581 = I.\omega^2.C_S = I \times (62.84)^2 \times 0.02 = 79\ I$$

$$\therefore \quad I = 2581/79 = 32.7 \text{ kg-m}^2 \textbf{ Ans.}$$

Dimensions of the flywheel rim

Let t = Thickness of the flywheel rim in metres,

b = Breadth of the flywheel rim in metres = $2\,t$ ... (Given)

D = Mean diameter of the flywheel in metres, and

v = Peripheral velocity of the flywheel in m/s.

We know that hoop stress (σ),

$$6 \times 10^6 = \rho.v^2 = 7250\ v^2 \quad \text{or} \quad v^2 = 6 \times 10^6/7250 = 827.6$$

$$\therefore \quad v = 28.8 \text{ m/s}$$

We know that $v = \pi DN/60$, or $D = v \times 60 / \pi N = 28.8 \times 60/\pi \times 600 = 0.92$ m

Now, let us find the mass (m) of the flywheel rim. Since the rim contributes 92% of the flywheel effect, therefore maximum fluctuation of energy of rim,

$$\Delta E_{rim} = 0.92 \times \Delta E = 0.92 \times 2581 = 2375 \text{ N-m}$$

We know that maximum fluctuation of energy of rim (ΔE_{rim}),

$$2375 = m.v^2.C_S = m \times (28.8)^2 \times 0.02 = 16.6\ m$$

$$\therefore \quad m = 2375/16.6 = 143 \text{ kg}$$

Also m = Volume × density = $\pi\, D.A.\rho = \pi\, D.b.t.\rho$

$$\therefore \quad 143 = \pi \times 0.92 \times 2\,t \times t \times 7250 = 41\,914\ t^2$$

$$t^2 = 143 / 41\,914 = 0.0034 \text{ m}^2$$

or $t = 0.0584$ m = 58.4 mm **Ans.**

and $b = 2\,t = 116.8$ mm **Ans.**

Example 16.17. *The turning moment diagram of a four stroke engine may be assumed for the sake of simplicity to be represented by four triangles in each stroke. The areas of these triangles are as follows:*

…ction stroke = 5 × 10^{-5} m^2; Compression stroke = 21 × 10^{-5} m^2; Expansion stroke = …10^{-5} m^2; Exhaust stroke = 8 × 10^{-5} m^2.

All the areas excepting expansion stroke are negative. Each m^2 of area represents 14 MN-m … work.

Assuming the resisting torque to be constant, determine the moment of inertia of the flywheel … the speed between 98 r.p.m. and 102 r.p.m. Also find the size of a rim-type flywheel based on … material criterion, given that density of flywheel material is 8150 kg/m^3; the allowable … stress of the flywheel material is 7.5 MPa. The rim cross-section is rectangular, one side … four times the length of the other.

Solution. Given: $a_1 = 5 \times 10^{-5}$ m²; $a_2 = 21 \times 10^{-5}$ m²; $a_3 = 85 \times 10^{-5}$ m²; $a_4 = 8 \times 10^{-5}$ m²; … = 98 r.p.m.; N_1 = 102 r.p.m.; ρ = 8150 kg/m³; σ = 7.5 MPa = 7.5 × 10^6 N/m²

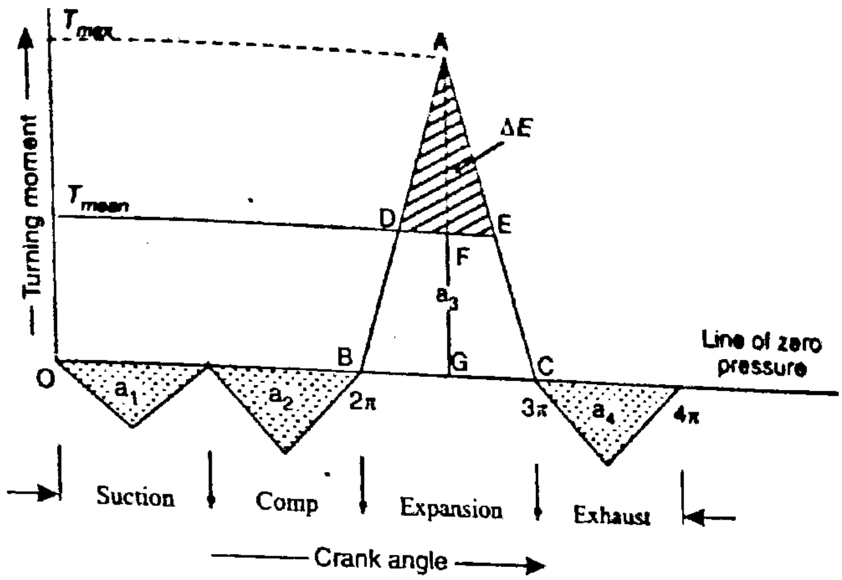


Fig. 16.20

The turning moment-crank angle diagram for a four stroke engine is shown in Fig. 16.20. The areas below the zero line of pressure are taken as negative while the areas above the zero line of pressure are taken as positive.

∴ Net area $= a_3 - (a_1 + a_2 + a_4)$

$= 85 \times 10^5 - (5 \times 10^5 + 21 \times 10^{-5} + 8 \times 10^{-5}) = 51 \times 10^{-5}$ m²

Since 1m² = 14 MN-m = 14 × 10^6 N-m of work, therefore

Net work done per cycle

$$= 51 \times 10^{-5} \times 14 \times 10^6 = 7140 \text{ N-m} \quad ...(i)$$

We also know that work done per cycle

$$= T_{mean} \times 4\pi \text{ N-m} \quad ...(ii)$$

From equations (*i*) and (*ii*),

$$T_{mean} = FG = 7140 / 4\pi = 568 \text{ N-m}$$

Work done during expansion stroke

$$= a_3 \times \text{Work scale} = 85 \times 10^{-5} \times 14 \times 10^6 = 11\,900 \text{ N-m} \quad ...(iii)$$

Also, work done during expansion stroke

$$= \frac{1}{2} \times BC \times AG = \frac{1}{2} \times \pi \times AG = 1.571\ AG$$

From equations (*iii*) and (*iv*),

$$AG = 11\,900/1.571 = 7575 \text{ N-m}$$

$\therefore$ Excess torque $= AF = AG - FG = 7575 - 568 = 7007$ N-m

Now from similar triangles *ADE* and *ABC*,

$$\frac{DE}{BC} = \frac{AF}{AG} \quad \text{or} \quad DE = \frac{AF}{AG} \times BC = \frac{7007}{7575} \times \pi = 2.9 \text{ rad}$$

We know that maximum fluctuation of energy,

$$\Delta E = \text{Area of } \Delta\, ADE = \frac{1}{2} \times DE \times AF$$

$$= \frac{1}{2} \times 2.9 \times 7007 = 10\,160 \text{ N-m}$$

Moment of Inertia of the flywheel

Let I = Moment of inertia of the flywheel in kg-m².

We know that mean speed during the cycle

$$N = \frac{N_1 + N_2}{2} = \frac{102 + 98}{2} = 100 \text{ r.p.m.}$$

$\therefore$ Corresponding angular mean speed,

$$\omega = 2\pi N / 60 = 2\pi \times 100/60 = 10.47 \text{ rad/s}$$

and coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = \frac{102 - 98}{100} = 0.04$$

We know that maximum fluctuation of energy (ΔE),

$$10\,160 = I.\omega^2.C_S = I\,(10.47)^2 \times 0.04 = 4.385\ I$$

$\therefore$ $I = 10160 / 4.385 = 2317$ kg-m² **Ans.**

Size of flywheel

Let t = Thickness of the flywheel rim in metres,

b = Width of the flywheel rim in metres = $4\,t$ ...(Given)

D = Mean diameter of the flywheel in metres, and

v = Peripheral velocity of the flywheel in m/s.

We know that hoop stress (σ),

$$7.5 \times 10^6 = \rho\,.\ v^2 = 8150\ v^2$$

$$\therefore \quad v^2 = \frac{7.5 \times 10^6}{8150} = 920 \text{ or } v = 30.3 \text{ m/s}$$

and $v = \pi DN/60$ or $D = v \times 60/\pi N = 30.3 \times 60/\pi \times 100 = 5.786$ m

Now let us find the mass (m) of the flywheel rim. We know that maximum fluctuation of energy (ΔE),

$$10\,160 = m.v^2\, C_S = m \times (30.3)^2 \times 0.04 = 36.72\, m$$

$\therefore$ $$m = 10\,160/36.72 = 276.7 \text{ kg}$$

Also $$m = \text{Volume} \times \text{density} = \pi D \times A \times \rho = \pi D \times b \times t \times \rho$$

$$276.7 = \pi \times 5.786 \times 4t \times t \times 8150 = 592\,655\, t^2$$

$\therefore$ $$t^2 = 276.7/592\,655 = 4.67 \times 10^{-4} \text{ or } t = 0.0216 \text{ m} = 21.6 \text{ mm } \textbf{Ans.}$$

$$b = 4t = 4 \times 21.6 = 86.4 \text{ mm } \textbf{Ans.}$$

Example 16.18. *An otto cycle engine develops 50 kW at 150 r.p.m. with 75 explosions per minute. The change of speed from the commencement to the end of power stroke must not exceed 0.5% of mean on either side. Find the mean diameter of the flywheel and a suitable rim cross-section having width four times the depth so that the hoop stress does not exceed 4 MPa. Assume that the flywheel stores 16/15 times the energy stored by the rim and the work done during power stroke is 1.40 times the work done during the cycle. Density of rim material is 7200 kg/m³.*

Solution. Given : $P = 50$ kW $= 50 \times 10^3$ W; $N = 150$ r.p.m. or $\omega = 2\pi \times 150/60 = 15.71$ rad/s; $n = 75$; $\sigma = 4$ MPa $= 4 \times 10^6$ N/m²; $\rho = 7200$ kg/m³

First of all, let us find the mean torque (T_{mean}) transmitted by the engine or flywheel. We know that the power transmitted (P),

$$50 \times 10^3 = T_{mean} \times \omega = T_{mean} \times 15.71$$

$\therefore$ $$T_{mean} = 50 \times 10^3/15.71 = 3182.7 \text{ N-m}$$

Since the explosions per minute are equal to $N/2$, therefore, the engine is a four stroke cycle engine. The turning moment diagram of a four stroke engine is shown in Fig. 16.21.

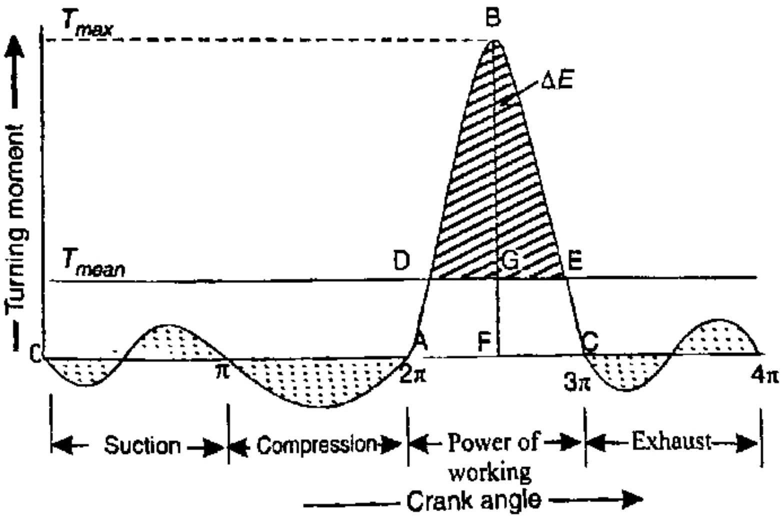


Fig. 16.21

We know that *work done per cycle

$$= T_{mean} \times \theta = 3182.7 \times 4\pi = 40\,000 \text{ N-m}$$

The work done per cycle for a four stroke engine is also given by

$$\text{Work done per cycle} = \frac{P \times 60}{\text{Number of explosions/min}} = \frac{P \times 60}{n} = \frac{50 \times 10^3 \times 60}{75} = 40000 \text{ N-m}$$

∴ Workdone during power or working stroke

$= 1.4 \times$ work done per cycle ...(Given)

$= 1.4 \times 40\,000 = 56\,000$ N-m ...(i)

The workdone during power stroke is shown by a triangle ABC in Fig. 16.20, in which base $AC = \pi$ radians and height $BF = T_{max}$.

∴ Work done during working stroke

$$= \frac{1}{2} \times \pi \times T_{max} = 1.571\, T_{max} \quad \ldots (ii)$$

From equations (i) and (ii), we have

$$T_{max} = 56\,000/1.571 = 35\,646 \text{ N-m}$$

We know that the excess torque,

$$T_{excess} = BG = BF - FG = T_{max} - T_{mean} = 35\,646 - 3182.7 = 32\,463.3 \text{ N-m}$$

Now, from similar triangles BDE and ABC,

$$\frac{DE}{AC} = \frac{BG}{BF} \quad \text{or} \quad DE = \frac{BG}{BF} \times AC = \frac{32\,463.3}{35646} \times \pi = 0.9107\pi$$

We know that maximum fluctuation of energy,

$$\Delta E = \text{Area of triangle } BDE = \frac{1}{2} \times DE \times BG$$

$$= \frac{1}{2} \times 0.9107\pi \times 32\,463.3 = 46\,445 \text{ N-m}$$

Mean diameter of the flywheel

Let D = Mean diameter of the flywheel in metres, and

v = Peripheral velocity of the flywheel in m/s.

We know that hoop stress (σ),

$$4 \times 10^6 = \rho . v^2 = 7200\, v^2 \quad \text{or} \quad v^2 = 4 \times 10^6/7200 = 556$$

∴ $v = 23.58$ m/s

We know that $v = \pi DN/60$ or $D = v \times 60/N = 23.58 \times 60/\pi \times 150 = 3$ m **Ans.**

Cross-sectional dimensions of the rim

Let t = Thickness of the rim in metres, and

b = Width of the rim in metres $= 4\,t$ ...(Given)

∴ Cross-sectional area of the rim,

$$A = b \times t = 4\,t \times t = 4\,t^2$$

First of all, let us find the mass of the flywheel rim.

Let m = Mass of the flywheel rim in kg, and

E = Total energy of the flywheel in N-m.

Since the fluctuation of speed is 0.5% of the mean speed on either side, therefore total fluctuation of speed,

$$N_2 - N_1 = 1\% \text{ of mean speed} = 0.01\, N$$

and coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = 0.01$$

We know that the maximum fluctuation of energy (ΔE),

$$46\,445 = E \times 2C_S = E \times 2 \times 0.01 = 0.02\ E$$

$$\therefore \quad E = 46\,445/0.02 = 2322 \times 10^3 \text{ N-m}$$

Since the energy stored by the flywheel is $\frac{16}{15}$ times the energy stored by the rim, therefore, the energy of the rim,

$$E_{rim} = \frac{15}{16} E = \frac{15}{16} \times 232 \times 10^3 = 2177 \times 10^3 \text{ N-m}$$

We know that energy of the rim (E_{rim}),

$$2177 \times 10^3 = \frac{1}{2} \times m \times v^2 = m\,(23.58)^2 = 278\ m$$

$$\therefore \quad m = 2177 \times 10^3 / 278 = 7831 \text{ kg}$$

We also know that mass of the flywheel rim (m),

$$7831 = \pi D \times A \times \rho = \pi \times 3 \times 4t^2 \times 7200 = 271\,469 t^2$$

$$\therefore \quad t^2 = 7831/271\,469 = 0.0288 \text{ or } t = 0.17 \text{ m} = 170 \text{ mm } \textbf{Ans.}$$

and $$b = 4\,t = 4 \times 170 = 680 \text{ mm } \textbf{Ans.}$$

16.12. Flywheel in Punching Press

We have discussed in Art. 16.8 that the function of a flywheel in an engine is to reduce the fluctuations of speed, when the load on the crankshaft is constant and the input torque varies during the cycle. The flywheel can also be used to perform the same function when the torque is constant and the load varies during the cycle. Such an application is found in punching press or in a rivetting machine. A punching press is shown diagrammatically in Fig. 16.22. The crank is driven by a motor which supplies constant torque and the punch is at the position of the slider in a slider-crank mechanism. From Fig. 16.22, we see that the load acts only during the rotation of the crank from $\theta = \theta_1$ to $\theta = \theta_2$, when the actual punching takes place and the load is zero for the rest of the cycle. Unless a flywheel is used, the speed of the crankshaft will increase too much during the rotation of crank from $\theta = \theta_2$ to $\theta = 2\pi$ or $\theta = 0$ and again from $\theta = 0$ to $\theta = \theta_1$, because there is no load while input energy continues to be supplied. On the other hand, the drop in speed of the crankshaft is very large during the rotation of crank from

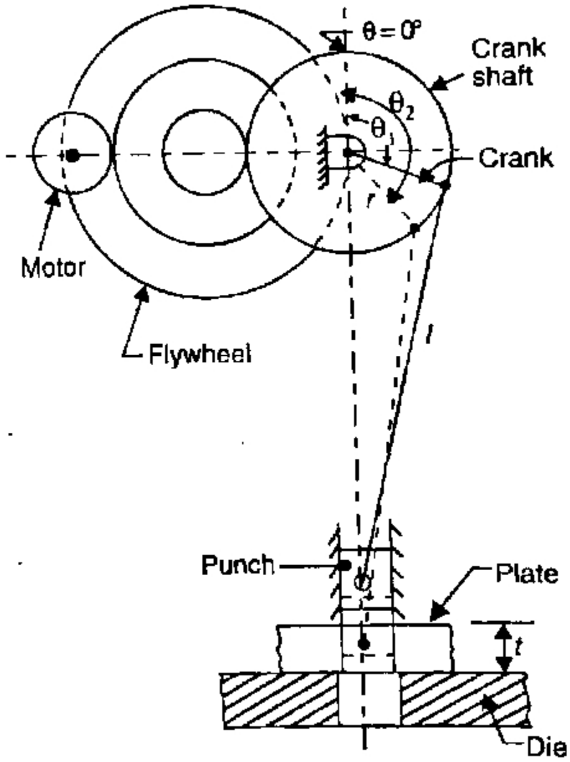


Fig. 16.22. Operation of flywheel in a punching press.

$\theta = \theta_1$ to $\theta = \theta_2$ due to much more load than the energy supplied. Thus the flywheel has to absorb excess energy available at one stage and has to make up the deficient energy at the other stage to keep the fluctuations of speed within permissible limits. This is done by choosing the suitable moment of inertia of the flywheel.

Punching press and flywheel.

Let E_1 be the energy required for punching a hole. This energy is determined by the size of the hole punched, the thickness of the material and the physical properties of the material.

Let d_1 = Diameter of the hole punched,

t_1 = Thickness of the plate, and

τ_u = Ultimate shear stress for the plate material.

$\therefore$ Maximum shear force required for punching,

F_S = Area sheared × Ultimate shear stress = $\pi d_1 . t_1 \tau_u$

It is assumed that as the hole is punched, the shear force decreases uniformly from maximum value to zero.

$\therefore$ Work done or energy required for punching a hole,

$$E_1 = \frac{1}{2} \times F_S \times t$$

Assuming one punching operation per revolution, the energy supplied to the shaft per revolution should also be equal to E_1. The energy supplied by the motor to the crankshaft during actual punching operation,

$$E_2 = E_1\left(\frac{\theta_2 - \theta_1}{2\pi}\right)$$

$\therefore$ Balance energy required for punching

$$= E_1 - E_2 = E_1 - E_1\left(\frac{\theta_2 - \theta_1}{2\pi}\right) = E_1\left(1 - \frac{\theta_2 - \theta_1}{2\pi}\right)$$

This energy is to be supplied by the flywheel by the decrease in its kinetic energy when its speed falls from maximum to minimum. Thus maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = E_1\left(1 - \frac{\theta_2 - \theta_1}{2\pi}\right)$$

The values of θ_1 and θ_2 may be determined only if the crank radius (r), length of connecting rod (l) and the relative position of the job with respect to the crankshaft axis are known. In the absence of relevant data, we assume that

$$\frac{\theta_2 - \theta_1}{2\pi} = \frac{t}{2s} = \frac{t}{4r}$$

t = Thickness of the material to be punched,

s = Stroke of the punch = 2 × Crank radius = $2r$.

By using the suitable relation for the maximum fluctuation of energy (ΔE) as discussed in the previous articles, we can find the mass and size of the flywheel.

Example 16.19. *A punching press is driven by a constant torque electric motor. The press is provided with a flywheel that rotates at maximum speed of 225 r.p.m. The radius of gyration of the flywheel is 0.5 m. The press punches 720 holes per hour; each punching operation takes 2 second and requires 15 kN-m of energy. Find the power of the motor and the minimum mass of the flywheel if speed of the same is not to fall below 200 r. p. m.*

Solution. Given N_1 = 225 r.p.m ; k = 0.5 m ; Hole punched = 720 per hr; E_1 = 15 kN-m = 15 × 10^3 N-m ; N_2 = 200 r.p.m.

Power of the motor

We know that the total energy required per second

= Energy required / hole × No. of holes / s

$= 15 \times 10^3 \times 720/3600 = 3000$ N-m/s

∴ Power of the motor = 3000 W = 3 kW **Ans.** (∵ 1 N-m/s = 1 W)

Minimum mass of the flywheel

Let m = Minimum mass of the flywheel.

Since each punching operation takes 2 seconds, therefore energy supplied by the motor in 2 seconds,

$$E_2 = 3000 \times 2 = 6000 \text{ N-m}$$

∴ Energy to be supplied by the flywheel during punching or maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = 15 \times 10^3 - 6000 = 9000 \text{ N-m}$$

Mean speed of the flywheel,

$$N = \frac{N_1 + N_2}{2} = \frac{225 + 200}{2} = 212.5 \text{ r.p.m}$$

We know that maximum fluctuation of energy (ΔE),

$$9000 = \frac{\pi^2}{900} \times m.k^2.N(N_1 - N_2)$$

$$= \frac{\pi^2}{900} \times m \times (0.5)^2 \times 212.5 \times (225 - 200) = 14.565\, m$$

∴ $m = 9000/14.565 = 618$ kg **Ans.**

Example 16.20. *A machine punching 38 mm holes in 32 mm thick plate requires 7 N-m of energy per sq. mm of sheared area, and punches one hole in every 10 seconds. Calculate the power of the motor required. The mean speed of the flywheel is 25 metres per second. The punch has a stroke of 100 mm.*

Find the mass of the flywheel required, if the total fluctuation of speed is not to exceed 3% of the mean speed. Assume that the motor supplies energy to the machine at uniform rate.

Solution. Given : d = 38 mm ; t = 32 mm ; E_1 = 7 N-m/mm^2 of sheared area ; v = 25 m/s ; l = 100 mm ; $v_1 - v_2 = 3\%\, v = 0.03\, v$

Power of the motor required

We know that sheared area,

$$A = \pi d.t = \pi \times 38 \times 32 = 3820 \text{ mm}^2$$

Since the energy required to punch a hole is 7 N-m/mm² of sheared area, therefore total energy required per hole,

$$E_1 = 7 \times 3820 = 26\,740 \text{ N-m}$$

Also the time required to punch a hole is 10 second, therefore energy required for punching work per second

$$= 26\,740/10 = 2674 \text{ N-m/s}$$

∴ Power of the motor required

$$= 2674 \text{ W} = 2.674 \text{ kW } \textbf{Ans.}$$

Mass of the flywheel required

Let m = Mass of the flywheel in kg.

Since the stroke of the punch is 100 mm and it punches one hole in every 10 seconds, therefore the time required to punch a hole in a 32 mm thick plate

$$= \frac{10}{2 \times 100} \times 32 = 1.6 \text{ s}$$

∴ Energy supplied by the motor in 1.6 seconds,

$$E_2 = 2674 \times 1.6 = 4278 \text{ N-m}$$

Energy to be supplied by the flywheel during punching or the maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = 26\,740 - 4278 = 22\,462 \text{ N-m}$$

Coefficient of fluctuation of speed,

$$C_S = \frac{v_1 - v_2}{v} = 0.03$$

We know that maximum fluctuation of energy (ΔE),

$$22\,462 = m.v^2.C_S = m \times (25)^2 \times 0.03 = 18.75\, m$$

$$\therefore \quad m = 22\,462 / 18.75 = 1198 \text{ kg } \textbf{Ans.}$$

Note : The value of maximum fluctuation of energy (ΔE) may also be determined as discussed in Art. 16.12. We know that energy required for one punch,

$$E_1 = 26\,740 \text{ N-m}$$

and

$$\Delta E = \left(1 - \frac{\theta_2 - \theta_1}{2\pi}\right) = E_1\left(1 - \frac{t}{2s}\right) \quad \ldots\ldots\left(\because \frac{\theta_2 - \theta_1}{2\pi} = \frac{t}{2s}\right)$$

$$= 26\,740\left[1 - \frac{32}{2 \times 100}\right] = 22\,462 \text{ N-m}$$

Example 16.21. *A riveting machine is driven by a constant torque 3 kW motor. The moving parts including the flywheel are equivalent to 150 kg at 0.6 m radius. One riveting operation takes 1 second and absorbs 10 000 N-m of energy. The speed of the flywheel is 300 r.p.m. before riveting. Find the speed immediately after riveting. How many rivets can be closed per minute?*

Solution. Given : P = 3 kW ; m = 150 kg ; k = 0.6 m ; N_1 = 300 r.p.m. or $\omega_1 = 2\pi \times 300/60 = 31.42$ rad/s

Speed of the flywheel immediately after riveting

Let ω_2 = Angular speed of the flywheel immediately after riveting.

We know that energy supplied by the motor,

$$E_2 = 3\text{kW} = 3000\text{ W} = 3000\text{ N-m/s} \qquad (\because 1\text{ W} = 1\text{ N-m/s})$$

But energy absorbed during one riveting operation which takes 1 second,

$$E_1 = 10\,000\text{ N-m}$$

∴ Energy to be supplied by the flywheel for each riveting operation per second or the maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = 10\,000 - 3000 = 7000\text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$7000 = \frac{1}{2} \times m.k^2\left[(\omega_1)^2 - (\omega_2)^2\right] = \frac{1}{2} \times 150 \times (0.6)^2 \times \left[(31.42)^2 - (\omega_2)^2\right]$$

$$= 27\left[987.2 - (\omega_2)^2\right]$$

∴ $$(\omega_2)^2 = 987.2 - 7000/27 = 728 \text{ or } \omega_2 = 26.98\text{ rad/s}$$

Corresponding speed in r.p.m.,

$$N_2 = 26.98 \times 60/2\pi = 257.6\text{ r.p.m.}$$ **Ans.**

Number of rivets that can be closed per minute

Since the energy absorbed by each riveting operation which takes 1 second is 10 000 N-m, therefore, number of rivets that can be closed per minute,

$$= \frac{E_2}{E_1} \times 60 = \frac{3000}{10\,000} \times 60 = 18 \text{ rivets}$$ **Ans.**

Example 16.22. *A punching press is required to punch 40 mm diameter holes in a plate of 15 mm thickness at the rate of 30 holes per minute. It requires 6 N-m of energy per mm² of sheared area. If the punching takes 1/10 of a second and the r.p.m. of the flywheel varies from 160 to 140, determine the mass of the flywheel having radius of gyration of 1 metre.*

Solution. Given: $d = 40$ mm; $t = 15$ mm; No. of holes = 30 per min.; Energy required = 6 N-m/mm²; Time = 1/10 s = 0.1 s; $N_1 = 160$ r.p.m.; $N_2 = 140$ r.p.m.; $k = 1$m

We know that sheared area per hole

$$= \pi\, d.t = \pi \times 40 \times 15 = 1885\text{ mm}^2$$

∴ Energy required to punch a hole,

$$E_1 = 6 \times 1885 = 11\,310\text{ N-m}$$

and energy required for punching work per second

$$= \text{Energy required per hole} \times \text{No. of holes per second}$$

$$= 11\,310 \times 30/60 = 5655\text{ N-m/s}$$

Since the punching takes 1/10 of a second, therefore, energy supplied by the motor in 1/10 second,

$$E_2 = 5655 \times 1/10 = 565.5\text{ N-m}$$

∴ Energy to be supplied by the flywheel during punching a hole or maximum fluctuation of energy of the flywheel,

$$\Delta E = E_1 - E_2 = 11\,310 - 565.5 = 10\,744.5\text{ N-m}$$

Mean speed of the flywheel,

$$N = \frac{N_1 + N_2}{2} = \frac{160 + 140}{2} = 150 \text{ r.p.m.}$$

We know that maximum fluctuation of energy (ΔE),

$$10\ 744.5 = \frac{\pi^2}{900} \times m.k^2\, N(N_1 - N_2)$$

$$= 0.011 \times m \times 1^2 \times 150\,(160 - 140) = 33m$$

$\therefore$ $m = 10744.5 / 33 = 327$ kg **Ans.**

Example 16.23. *A punching machine makes 25 working strokes per minute and is capable of punching 25 mm diameter holes in 18 mm thick steel plates having an ultimate shear strength 300 MPa. The punching operation takes place during 1/10th of a revolution of the crankshaft.*

Estimate the power needed for the driving motor, assuming a mechanical efficiency of 95 percent. Determine suitable dimensions for the rim cross-section of the flywheel, having width equal to twice thickness. The flywheel is to revolve at 9 times the speed of the crankshaft. The permissible coefficient of fluctuation of speed is 0.1.

The flywheel is to be made of cast iron having a working stress (tensile) of 6 MPa and density of 7250 kg/m³. The diameter of the flywheel must not exceed 1.4 m owing to space restrictions. The hub and the spokes may be assumed to provide 5% of the rotational inertia of the wheel.

Solution. Given : $n = 25$; $d_1 = 25$ mm $= 0.025$ m; $t_1 = 18$ mm $= 0.018$ m ; $\tau_u = 300$ MPa $= 300 \times 10^6$ N/m²; $\eta_m = 95\% = 0.95$; $C_S = 0.1$; $\sigma = 6$ MPa $= 6 \times 10^6$ N/m²; $\rho = 7250$ kg/m³; $D = 1.4$ m or $R = 0.7$ m

Power needed for the driving motor

We know that the area of plate sheared ,

$$A_S = \pi d_1 \times t_1 = \pi \times 0.025 \times 0.018 = 1414 \times 10^{-6} \text{ m}^2$$

$\therefore$ Maximum shearing force required for punching,

$$F_S = A_S \times \tau_u = 1414 \times 10^{-6} \times 300 \times 10^6 = 424\ 200 \text{ N}$$

and energy required per stroke

$$= \text{Average shear force} \times \text{Thickness of plate}$$

$$= \frac{1}{2} \times F_S \times t_1 = \frac{1}{2} \times 424\ 200 \times 0.018 = 3817.8 \text{ N-m}$$

$\therefore$ Energy required per min

$$= \text{Energy/stroke} \times \text{No. of working strokes/min}$$

$$= 3817.8 \times 25 = 95\ 450 \text{ N-m}$$

We know that the power needed for the driving motor

$$= \frac{\text{Energy required per min}}{60 \times \eta_m} = \frac{95\ 450}{60 \times 0.95} = 1675 \text{ W} = 1.675 \text{ kW} \text{ Ans.}$$

Dimensions for the rim cross-section

Let t = Thickness of rim in metres, and

b = Width of rim in metres $= 2t$ ... (Given)

$\therefore$ Cross-sectional area of rim,

$$A = b \times t = 2t \times t = 2t^2$$

Since the punching operation takes place (*i.e.* energy is consumed) during 1/10th of a revolution of the crankshaft, therefore during 9/10th of the revolution of a crankshaft, the energy is stored in the flywheel.

∴ Maximum fluctuation of energy,

$$\Delta E = \frac{9}{10} \times \text{Energy/stroke} = \frac{9}{10} \times 3817.8 = 3436 \text{ N-m}$$

Let

$$m = \text{Mass of the flywheel in kg.}$$

Since the hub and the spokes provide 5% of the rotational inertia of the wheel, therefore the maximum fluctuation of energy provided by the flywheel by the rim will be 95%.

∴ Maximum fluctuation of energy provided by the rim,

$$\Delta E_{rim} = 0.95 \times \Delta E = 0.95 \times 3436 = 3264 \text{ N-m}$$

Since the flywheel is to revolve at 9 times the speed of the crankshaft and there are 25 working strokes per minute, therefore, mean speed of the flywheel,

$$N = 9 \times 25 = 225 \text{ r.p.m.}$$

and mean angular speed,

$$\omega = 2\pi \times 225/60 = 23.56 \text{ rad/s}$$

We know that maximum fluctuation of energy (ΔE_{rim}),

$$3264 = m.R^2.\omega^2.C_S = m \times (0.7)^2 \times (23.56)^2 \times 0.1 = 27.2m$$

$$\therefore \quad m = 3264/27.2 = 120 \text{ kg}$$

We also know that mass of the flywheel (m),

$$120 = \pi D \times A \times \rho = \pi \times 1.4 \times 2t^2 \times 7250 = 63\,782 t^2$$

$$\therefore \quad t^2 = 120/63\,782 = 0.001\,88 \text{ or } t = 0.044 \text{ m} = 44 \text{ mm } \textbf{Ans.}$$

and

$$b = 2\,t = 2 \times 44 = 88 \text{ mm } \textbf{Ans.}$$

EXERCISES

1. An engine flywheel has a mass of 6.5 tonnes and the radius of gyration is 2 m. If the maximum and minimum speeds are 120 r. p. m. and 118 r. p. m. respectively, find maximum fluctuation of energy. **[Ans. 67. 875 kN-m]**

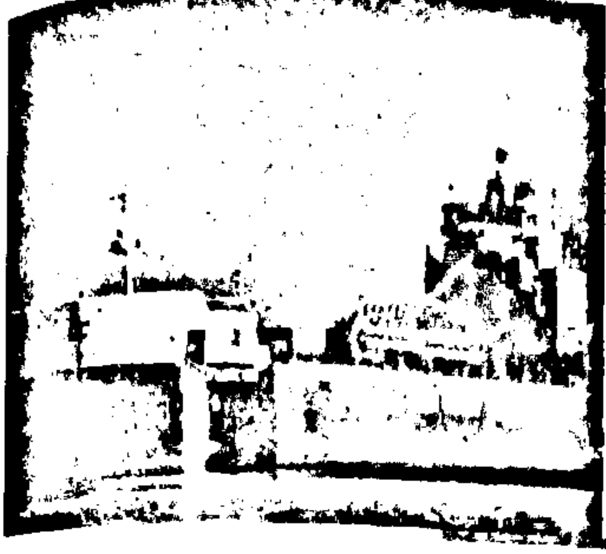
2. A vertical double acting steam engine develops 75 kW at 250 r.p.m. The maximum fluctuation of energy is 30 per cent of the work done per stroke. The maximum and minimum speeds are not to vary more than 1 per cent on either side of the mean speed. Find the mass of the flywheel required, if the radius of gyration is 0.6 m. **[Ans. 547 kg]**

3. In a turning moment diagram, the areas above and below the mean torque line taken in order are 4400, 1150, 1300 and 4550 mm^2 respectively. The scales of the turning moment diagram are:

   Turning moment, 1 mm = 100 N-m ; Crank angle, 1 mm = 1°

   Find the mass of the flywheel required to keep the speed between 297 and 303 r.p.m., if the radius of gyration is 0.525 m. **[Ans. 417 kg]**

4. The turning moment diagram for a multicylinder engine has been drawn to a scale of 1 mm = 4500 N-m vertically and 1 mm = 2.4° horizontally. The intercepted areas between output torque curve and mean resistance line taken in order from one end are 342, 23, 245, 303, 115, 232, 227, 164 mm^2, when the engine is running at 150 r.p.m. If the mass of the flywheel is 1000 kg and the total fluctuation of speed does not exceed 3% of the mean speed, find the minimum value of the radius of gyration. **[Ans. 1.034 m]**



21

Balancing of Rotating Masses

Features

1. *Introduction.*
2. *Balancing of Rotating Masses.*
3. *Balancing of a Single Rotating Mass By a Single Mass Rotating in the Same Plane.*
4. *Balancing of a Single Rotating Mass By Two Masses Rotating in Different Planes.*
5. *Balancing of Several Masses Rotating in the Same Plane.*
6. *Balancing of Several Masses Rotating in Different Planes.*

21.1. Introduction

The high speed of engines and other machines is a common phenomenon now-a-days. It is, therefore, very essential that all the rotating and reciprocating parts should be completely balanced as far as possible. If these parts are not properly balanced, the dynamic forces are set up. These forces not only increase the loads on bearings and stresses in the various members, but also produce unpleasant and even dangerous vibrations. In this chapter we shall discuss the balancing of unbalanced forces caused by rotating masses, in order to minimise pressure on the main bearings when an engine is running.

21.2. Balancing of Rotating Masses

We have already discussed, that whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it. In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal force of both the masses are made to be equal and opposite. The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass, is called ***balancing of rotating masses.***

The following cases are important from the subject point of view:

1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of a single rotating mass by two masses rotating in different planes.
3. Balancing of different masses rotating in the same plane.
4. Balancing of different masses rotating in different planes.

We shall now discuss these cases, in detail, in the following pages.

21.3. Balancing of a Single Rotating Mass By a Single Mass Rotating in the Same Plane

Consider a disturbing mass m_1 attached to a shaft rotating at ω rad/s as shown in Fig. 21.1. Let r_1 be the radius of rotation of the mass m_1 (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of the mass m_1).

We know that the centrifugal force exerted by the mass m_1 on the shaft,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1 \quad \ldots (i)$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft. In order to counteract the effect of this force, a balancing mass (m_2) may be attached in the same plane of rotation as that of disturbing mass (m_1) such that the centrifugal forces due to the two masses are equal and opposite.

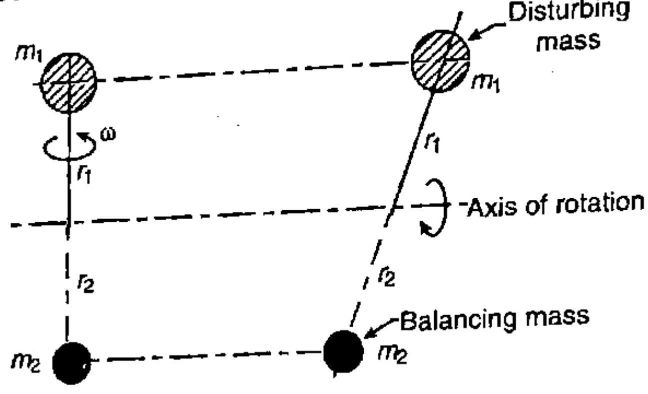


Fig. 21.1. Balancing of a single rotating mass by a single mass rotating in the same plane.

Let r_2 = Radius of rotation of the balancing mass m_2 (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of mass m_2).

$\therefore$ Centrifugal force due to mass m_2,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2 \quad \ldots (ii)$$

Equating equations (*i*) and (*ii*),

$$m_1 . \omega^2 \cdot r_1 = m_2 \cdot \omega^2 \cdot r_2 \quad \text{or} \quad m_1 \cdot r_1 = m_2 \cdot r_2$$

Notes : 1. The product $m_2.r_2$ may be split up in any convenient way. But the radius of rotation of the balancing mass (m_2) is generally made large in order to reduce the balancing mass m_2.

2. The centrifugal forces are proportional to the product of the mass and radius of rotation of respective masses, because ω^2 is same for each mass.

21.4. Balancing of a Single Rotating Mass By Two Masses Rotating in Different Planes

We have discussed in the previous article that by introducing a single balancing mass in the same plane of rotation as that of disturbing mass, the centrifugal forces are balanced. In other words, the two forces are equal in magnitude and opposite in direction. But this type of arrangement for balancing gives rise to a couple which tends to rock the shaft in its bearings. Therefore in order to put the system in complete balance, two balancing masses are placed in two different planes, parallel to the plane of rotation of the disturbing mass, in such a way that they satisfy the following two conditions of equilibrium.

1. The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the centre of the masses of the system must lie on the axis of rotation. This is the condition for ***static balancing***.

2. The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

The conditions (1) and (2) together give ***dynamic balancing***. The following two possibilities may arise while attaching the two balancing masses :

1. The plane of the disturbing mass may be in between the planes of the two balancing masses, and

2. The plane of the disturbing mass may lie on the left or right of the two planes containing the balancing masses.

We shall now discuss both the above cases one by one.

The picture shows a diesel engine. All diesel, petrol and steam engines have reciprocating and rotating masses inside them which need to be balanced.

1. *When the plane of the disturbing mass lies in between the planes of the two balancing masses*

Consider a disturbing mass m lying in a plane A to be balanced by two rotating masses m_1 and m_2 lying in two different planes L and M as shown in Fig. 21.2. Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, L and M respectively.

Let l_1 = Distance between the planes A and L,

l_2 = Distance between the planes A and M, and

l = Distance between the planes L and M.

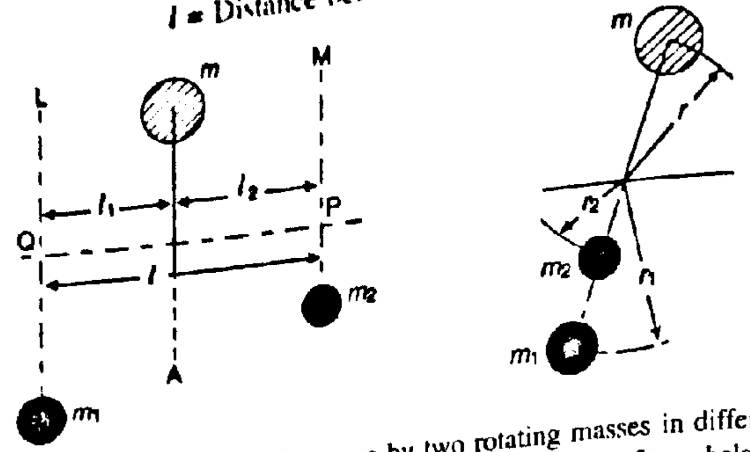


Fig. 21.2. Balancing of a single rotating mass by two rotating masses in different planes when the plane of single rotating mass lies in between the planes of two balancing masses.

We know that the centrifugal force exerted by the mass m in the plane A,

$$F_C = m \cdot \omega^2 \cdot r$$

Similarly, the centrifugal force exerted by the mass m_1 in the plane L,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1$$

and, the centrifugal force exerted by the mass m_2 in the plane M,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2$$

Since the net force acting on the shaft must be equal to zero, therefore the centrifugal force on the disturbing mass must be equal to the sum of the centrifugal forces on the balancing masses, therefore

$$F_C = F_{C1} + F_{C2} \quad \text{or} \quad m \cdot \omega^2 \cdot r = m_1 . \omega^2 \cdot r_1 + m_2 \cdot \omega^2 \cdot r_2$$

$$\therefore \quad m \cdot r = m_1 \cdot r_1 + m_2 \cdot r_2 \qquad \ldots (i)$$

Now in order to find the magnitude of balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2 \quad \text{or} \quad m_1 \cdot \omega^2 \cdot r_1 \times l = m \cdot \omega^2 \cdot r \times l_2$$

$$\therefore \quad m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \quad \text{or} \quad m_1 \cdot r_1 = m \cdot r \times \frac{l_2}{l} \qquad \ldots (ii)$$

Similarly, in order to find the balancing force in plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{C2} \times l = F_C \times l_1 \quad \text{or} \quad m_2 \cdot \omega^2 \cdot r_2 \times l = m \cdot \omega^2 \cdot r \times l_1$$

$$\therefore \quad m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1 \quad \text{or} \quad m_2 \cdot r_2 = m \cdot r \times \frac{l_1}{l} \qquad \ldots (iii)$$

It may be noted that equation (i) represents the condition for static balance, but in order to achieve dynamic balance, equations (ii) or (iii) must also be satisfied.

2. When the plane of the disturbing mass lies on one end of the planes of the balancing masses

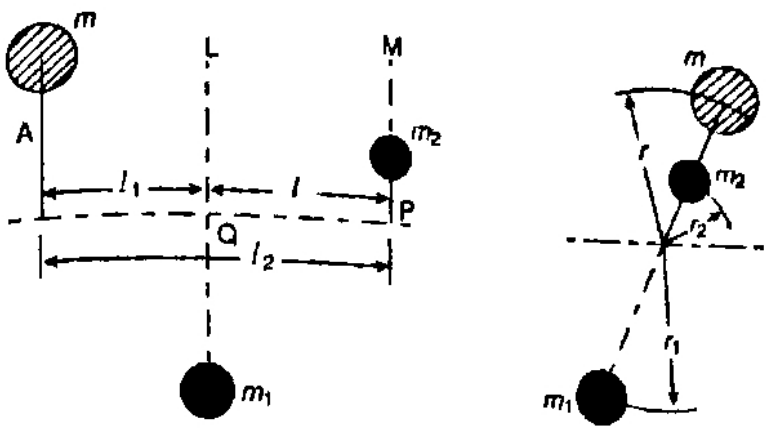


Fig. 21.3. Balancing of a single rotating mass by two rotating masses in different planes, when the plane of single rotating mass lies at one end of the planes of balancing masses.

In this case, the mass m lies in the plane A and the balancing masses lie in the planes L and M, as shown in Fig. 21.3. As discussed above, the following conditions must be satisfied in order to balance the system, *i.e.*

$$F_C + F_{C2} = F_{C1} \quad \text{or} \quad m \cdot \omega^2 \cdot r + m_2 \cdot \omega^2 \cdot r_2 = m_1 \cdot \omega^2 \cdot r_1$$

$$\therefore \quad m \cdot r + m_2 \cdot r_2 = m_1 . r_1 \qquad \ldots (iv)$$

Now, to find the balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2 \quad \text{or} \quad m_1 \cdot \omega^2 \cdot r_1 \times l = m \cdot \omega^2 \cdot r \times l_2$$

$$\therefore \quad m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \quad \text{or} \quad m_1 \cdot r_1 = m \cdot r \times \frac{l_2}{l} \qquad \ldots (v)$$

. . . [Same as equation (ii)]

Similarly, to find the balancing force in the plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{C2} \times l = F_C \times l_1 \quad \text{or} \quad m_2 \cdot \omega^2 \cdot r_2 \times l = m \cdot \omega^2 \cdot r \times l_1$$

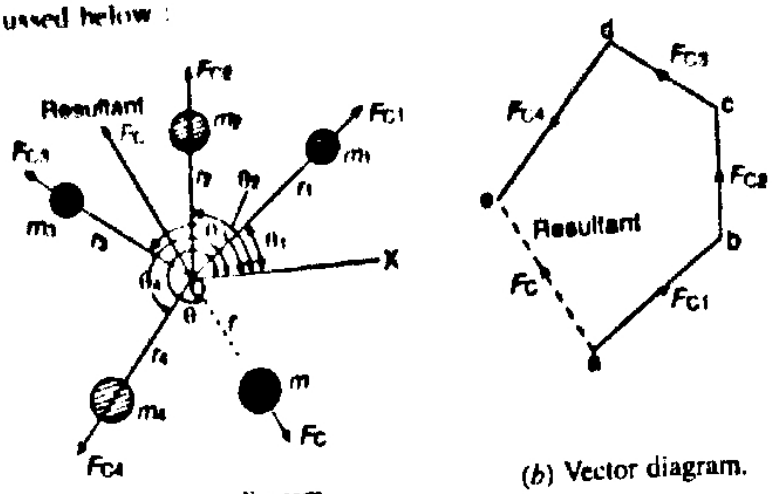
$$m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1 \quad \text{or} \quad m_2 \cdot r_2 = m \cdot r \times \frac{l_1}{l} \qquad \ldots (vi)$$

. . . [Same as equation (iii)]

21.5. Balancing of Several Masses Rotating in the Same Plane

Consider any number of masses (say four) of magnitude m_1, m_2, m_3 and m_4 at distances of r_1, r_2, r_3 and r_4 from the axis of the rotating shaft. Let $\theta_1, \theta_2, \theta_3$ and θ_4 be the angles of these masses with the horizontal line OX, as shown in Fig. 21.4 (*a*). Let these masses rotate about an axis through O and perpendicular to the plane of paper, with a constant angular velocity of ω rad/s.

The magnitude and position of the balancing mass may be found out analytically or graphically as discussed below :



(a) Space diagram.

(b) Vector diagram.

Fig. 21.4. Balancing of several masses rotating in the same plane.

1. ***Analytical method***

The magnitude and direction of the balancing mass may be obtained, analytically, as discussed below :

1. First of all, find out the centrifugal force* (or the product of the mass and its radius of rotation) exerted by each mass on the rotating shaft.

A car assembly line.

Note : This picture is given as additional information.

* Since ω^2 is same for each mass, therefore the magnitude of the centrifugal force for each mass is proportional to the product of the respective mass and its radius of rotation.

2. Resolve the centrifugal forces horizontally and vertically and find their sums, i.e. ΣH and ΣV. We know that

Sum of horizontal components of the centrifugal forces,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + \ldots\ldots$$

and sum of vertical components of the centrifugal forces,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + \ldots\ldots$$

3. Magnitude of the resultant centrifugal force,

$$F_C = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

4. If θ is the angle, which the resultant force makes with the horizontal, then

$$\tan\theta = \Sigma V / \Sigma H$$

5. The balancing force is then equal to the resultant force, but in ***opposite direction***.
6. Now find out the magnitude of the balancing mass, such that

$$F_C = m \cdot r$$

where m = Balancing mass, and

r = Its radius of rotation.

2. *Graphical method*

The magnitude and position of the balancing mass may also be obtained graphically as discussed below :

1. First of all, draw the space diagram with the positions of the several masses, as shown in Fig. 21.4 (*a*).
2. Find out the centrifugal force (or product of the mass and radius of rotation) exerted by each mass on the rotating shaft.
3. Now draw the vector diagram with the obtained centrifugal forces (or the product of the masses and their radii of rotation), such that *ab* represents the centrifugal force exerted by the mass m_1 (or $m_1.r_1$) in magnitude and direction to some suitable scale. Similarly, draw *bc*, *cd* and *de* to represent centrifugal forces of other masses m_2, m_3 and m_4 (or $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$).
4. Now, as per polygon law of forces, the closing side *ae* represents the resultant force in magnitude and direction, as shown in Fig. 21.4 (*b*).
5. The balancing force is, then, equal to the resultant force, but in ***opposite direction.***
6. Now find out the magnitude of the balancing mass (m) at a given radius of rotation (r), such that

$$m \cdot \omega^2 \cdot r = \text{Resultant centrifugal force}$$

or

$$m.r = \text{Resultant of } m_1.r_1,\ m_2.r_2,\ m_3.r_3 \text{ and } m_4.r_4$$

Example 21.1. *Four masses m_1, m_2, m_3 and m_4 are 200 kg, 300 kg, 240 kg and 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m and 0.3 m respectively and the angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m.*

Solution. Given : m_1 = 200 kg ; m_2 = 300 kg ; m_3 = 240 kg ; m_4 = 260 kg ; r_1 = 0.2 m ; r_2 = 0.15 m ; r_3 = 0.25 m ; r_4 = 0.3 m ; $\theta_1 = 0°$; $\theta_2 = 45°$; $\theta_3 = 45° + 75° = 120°$; $\theta_4 = 45° + 75° + 135° = 255°$; r = 0.2 m

Let m = Balancing mass, and

θ = The angle which the balancing mass makes with m_1

Since the magnitude of centrifugal forces are proportional to the product of each mass and its radius, therefore

$$m_1 \cdot r_1 = 200 \times 0.2 = 40 \text{ kg-m}$$

$$m_2 \cdot r_2 = 300 \times 0.15 = 45 \text{ kg-m}$$

$$m_3 \cdot r_3 = 240 \times 0.25 = 60 \text{ kg-m}$$

$$m_4 \cdot r_4 = 260 \times 0.3 = 78 \text{ kg-m}$$

The problem may, now, be solved either analytically or graphically. But we shall solve the problem by both the methods one by one.

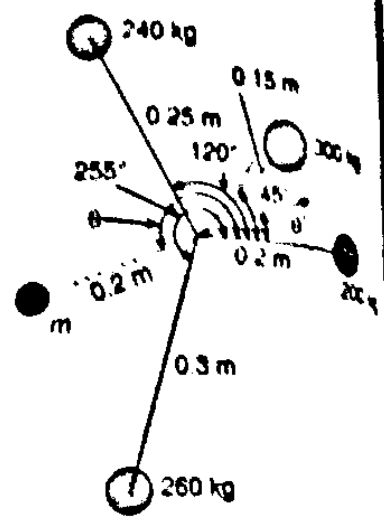


Fig. 21.5

1. *Analytical method*

The space diagram is shown in Fig. 21.5.

Resolving $m_1.r_1$, $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$ horizontally,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + m_3 \cdot r_3 \cos\theta_3 + m_4 \cdot r_4 \cos\theta_4$$

$$= 40\cos 0° + 45\cos 45° + 60\cos 120° + 78\cos 255°$$

$$= 40 + 31.8 - 30 - 20.2 = 21.6 \text{ kg-m}$$

Now resolving vertically,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + m_3 \cdot r_3 \sin\theta_3 + m_4 \cdot r_4 \sin\theta_4$$

$$= 40\sin 0° + 45\sin 45° + 60\sin 120° + 78\sin 255°$$

$$= 0 + 31.8 + 52 - 75.3 = 8.5 \text{ kg-m}$$

$\therefore$ Resultant, $R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(21.6)^2 + (8.5)^2} = 23.2$ kg-m

We know that

$$m \cdot r = R = 23.2 \quad \text{or} \quad m = 23.2 / r = 23.2 / 0.2 = 116 \text{ kg} \quad \textbf{Ans.}$$

and

$$\tan\theta' = \Sigma V / \Sigma H = 8.5 / 21.6 = 0.3935 \quad \text{or} \quad \theta' = 21.48°$$

Since θ' is the angle of the resultant R from the horizontal mass of 200 kg, therefore the angle of the balancing mass from the horizontal mass of 200 kg,

$$\theta = 180° + 21.48° = 201.48° \quad \textbf{Ans.}$$

2. *Graphical method*

The magnitude and the position of the balancing mass may also be found graphically as discussed below :

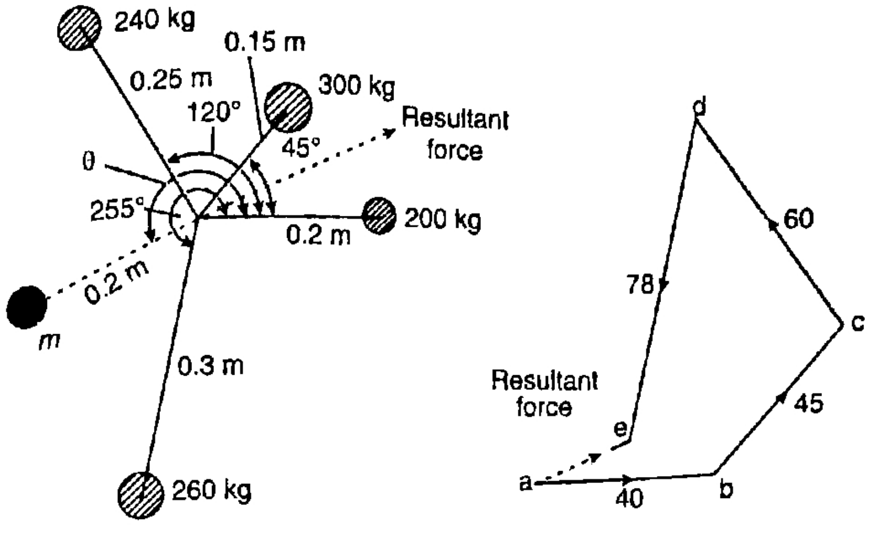
1. First of all, draw the space diagram showing the positions of all the given masses as shown in Fig 21.6 (*a*).
2. Since the centrifugal force of each mass is proportional to the product of the mass and radius, therefore

$$m_1.r_1 = 200 \times 0.2 = 40 \text{ kg-m}$$
$$m_2.r_2 = 300 \times 0.15 = 45 \text{ kg-m}$$

$m_3.r_3 = 240 \times 0.25 = 60$ kg-m

$m_4.r_4 = 260 \times 0.3 = 78$ kg-m

3. Now draw the vector diagram with the above values, to some suitable scale, as shown in Fig. 21.6 (*b*). The closing side of the polygon *ae* represents the resultant force. By measurement, we find that *ae* = 23 kg-m.



(*a*) Space diagram. (*b*) Vector diagram

Fig. 21.6

4. The balancing force is equal to the resultant force, but ***opposite*** in direction as shown in Fig. 21.6 (*a*). Since the balancing force is proportional to *m.r*, therefore

$$m \times 0.2 = \text{vector } ea = 23 \text{ kg-m} \quad \text{or} \quad m = 23/0.2 = \mathbf{115\ kg}$$ **Ans.**

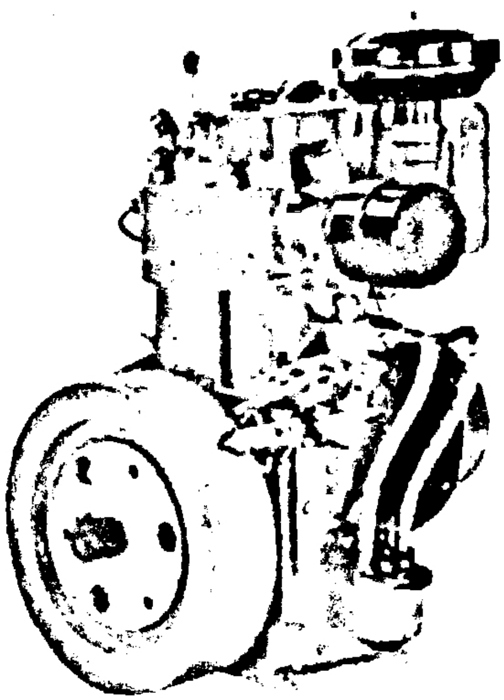
By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal mass of 200 kg,

$$\theta = 201°$$ **Ans.**

21.6. Balancing of Several Masses Rotating in Different Planes

When several masses revolve in different planes, they may be transferred to a ***reference plane*** (briefly written as *R.P.*), which may be defined as the plane passing through a point on the axis of rotation and perpendicular to it. The effect of transferring a revolving mass (in one plane) to a reference plane is to cause a force of magnitude equal to the centrifugal force of the revolving mass to act in the reference plane, together with a couple of magnitude equal to the product of the force and the distance between the plane of rotation and the reference plane. In order to have a complete balance of the several revolving masses in different planes, the following two conditions must be satisfied :

1. The forces in the reference plane must balance, *i.e.* the resultant force must be zero.
2. The couples about the reference plane must balance, *i.e.* the resultant couple must be zero.



Diesel engine.

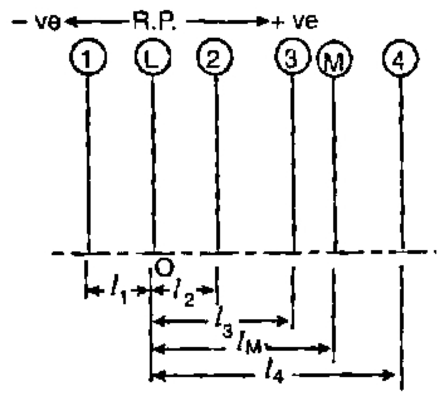
Let us now consider four masses m_1, m_2, m_3 and m_4 revolving in planes 1, 2, 3 and 4 respectively as shown in

Fig. 21.7 (*a*). The relative angular positions of these masses are shown in the end view [Fig. 21.7 (*b*)]. The magnitude of the balancing masses m_L and m_M in planes L and M may be obtained as discussed below :

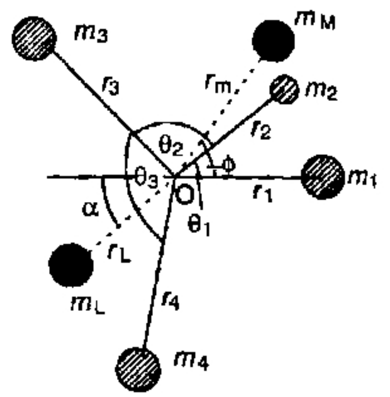
1. Take one of the planes, say L as the reference plane (*R.P.*). The distances of all the other planes to the left of the reference plane may be regarded as ***negative***, and those to the right as ***positive***.
2. Tabulate the data as shown in Table 21.1. The planes are tabulated in the same order in which they occur, reading from left to right.

Table 21.1

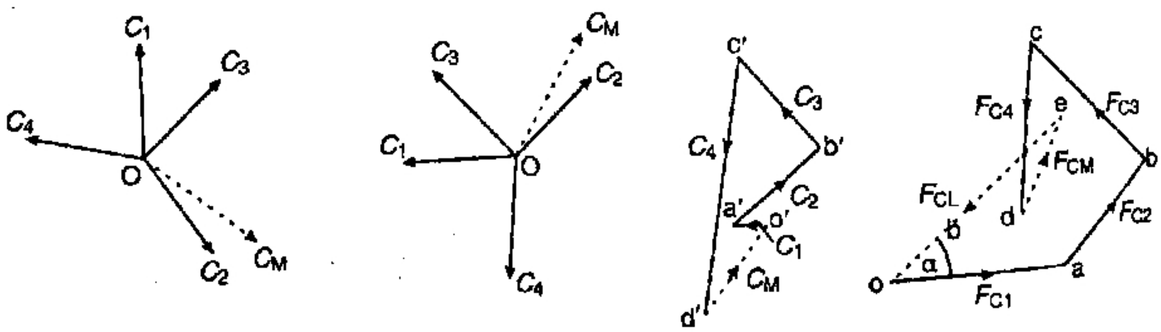
Plane (1)	*Mass (m)* (2)	*Radius(r)* (3)	*Cent.force* $\div \omega^2$ *(m.r)* (4)	*Distance from plane L (l)* (5)	*Couple* $\div \omega^2$ *(m.r.l)* (6)
1	m_1	r_1	$m_1.r_1$	$-l_1$	$-m_1.r_1.l_1$
L(R.P.)	m_L	r_L	$m_L.r_L$	0	0
2	m_2	r_2	$m_2.r_2$	l_2	$m_2.r_2.l_2$
3	m_3	r_3	$m_3.r_3$	l_3	$m_3.r_3.l_3$
M	m_M	r_M	$m_M.r_M$	l_M	$m_M.r_M.l_M$
4	m_4	r_4	$m_4.r_4$	l_4	$m_4.r_4.l_4$



(*a*) Position of planes of the masses.



(*b*) Angular position of the masses.



(*c*) Couple vector. (*d*) Couple vectors turned counter clockwise through a right angle. (*e*) Couple polygon. (*f*) Force polygon.

Fig. 21.7. Balancing of several masses rotating in different planes.

3. A couple may be represented by a vector drawn perpendicular to the plane of the couple. The couple C_1 introduced by transferring m_1 to the reference plane through O is propor-

tional to $m_1.r_1.l_1$ and acts in a plane through Om_1 and perpendicular to the paper. The vector representing this couple is drawn in the plane of the paper and perpendicular to Om_1 as shown by OC_1 in Fig. 21.7 (*c*). Similarly, the vectors OC_2, OC_3 and OC_4 are drawn perpendicular to Om_2, Om_3 and Om_4 respectively and in the plane of the paper.

4. The couple vectors as discussed above, are turned counter clockwise through a right angle for convenience of drawing as shown in Fig. 21.7 (*d*). We see that their relative positions remains unaffected. Now the vectors OC_2, OC_3 and OC_4 are parallel and in the same direction as Om_2, Om_3 and Om_4, while the vector OC_1 is parallel to Om_1 but in *opposite direction. Hence the ***couple vectors are drawn radially outwards for the masses on one side of the reference plane and radially inward for the masses on the other side of the reference plane.***

5. Now draw the couple polygon as shown in Fig. 21.7 (*e*). The vector $d'o'$ represents the balanced couple. Since the balanced couple C_M is proportional to $m_M.r_M.l_M$, therefore

$$C_M = m_M \cdot r_M \cdot l_M = \text{vector } d'o' \quad \text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M \cdot l_M}$$

From this expression, the value of the balancing mass m_M in the plane *M* may be obtained, and the angle of inclination ϕ of this mass may be measured from Fig. 21.7 (*b*).

6. Now draw the force polygon as shown in Fig. 21.7 (*f*). The vector *eo* (in the direction from *e* to *o*) represents the balanced force. Since the balanced force is proportional to $m_L.r_L$, therefore,

$$m_L \cdot r_L = \text{vector } eo \quad \text{or} \quad m_L = \frac{\text{vector } eo}{r_L}$$

From this expression, the value of the balancing mass m_L in the plane *L* may be obtained and the angle of inclination α of this mass with the horizontal may be measured from Fig. 21.7 (*b*).

Example 21.2. *A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45°, B to C 70° and C to D 120°. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.*

Solution. Given : m_A = 200 kg ; m_B = 300 kg ; m_C = 400 kg ; m_D = 200 kg ; r_A = 80 mm = 0.08m ; r_B = 70 mm = 0.07 m ; r_C = 60 mm = 0.06 m ; r_D = 80 mm = 0.08 m ; $r_X = r_Y$ = 100 mm = 0.1 m

Let m_X = Balancing mass placed in plane *X*, and

m_Y = Balancing mass placed in plane *Y*.

The position of planes and angular position of the masses (assuming the mass A as horizontal) are shown in Fig. 21.8 (*a*) and (*b*) respectively.

Assume the plane *X* as the reference plane (*R.P.*). The distances of the planes to the right of plane *X* are taken as + ve while the distances of the planes to the left of plane *X* are taken as – ve. The data may be tabulated as shown in Table 21.2.

* From Table 21.1 (column 6) we see that the couple is $-m_1.r_1.l_1$.

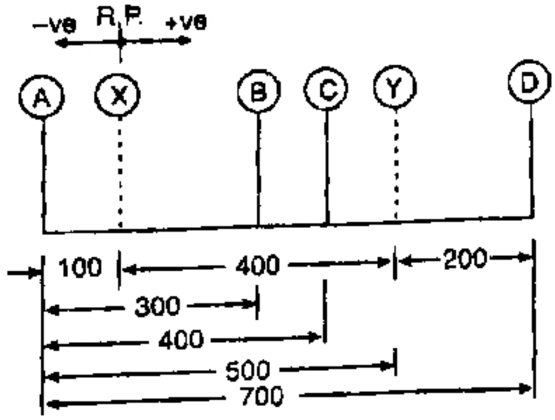
Table 21.2

Plane (1)	*Mass (m)* kg (2)	*Radius (r)* m (3)	*Cent.force* ÷ ω^2 *(m.r)* kg-m (4)	*Distance from Plane x(l)* m (5)	*Couple* ÷ ω^2 *(m.r.l)* kg-m² (6)
A	200	0.08	16	– 0.1	– 1.6
X(*R.P.*)	m_X	0.1	0.1 m_X	0	0
B	300	0.07	21	0.2	4.2
C	400	0.06	24	0.3	7.2
Y	m_Y	0.1	0.1 m_Y	0.4	0.04 m_Y
D	200	0.08	16	0.6	9.6

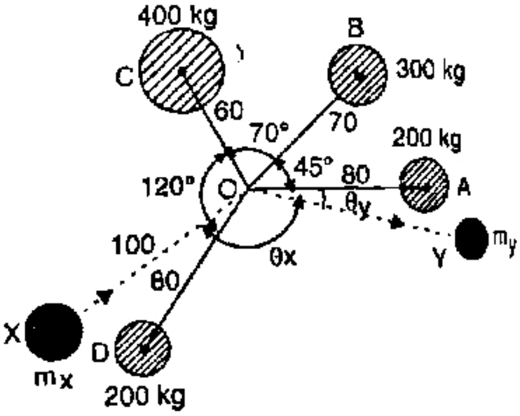
The balancing masses m_X and m_Y and their angular positions may be determined graphically as discussed below :

1. First of all, draw the couple polygon from the data given in Table 21.2 (column 6) as shown in Fig. 21.8 (*c*) to some suitable scale. The vector $d'o'$ represents the balanced couple. Since the balanced couple is proportional to 0.04 m_Y, therefore by measurement,

$$0.04\,m_Y = \text{vector } d'o' = 7.3 \text{ kg-m}^2 \qquad \text{or} \qquad m_Y = 182.5 \text{ kg} \textbf{ Ans.}$$

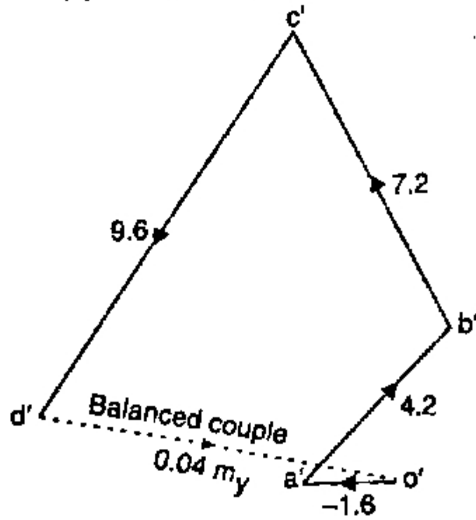


(*a*) Position of planes.

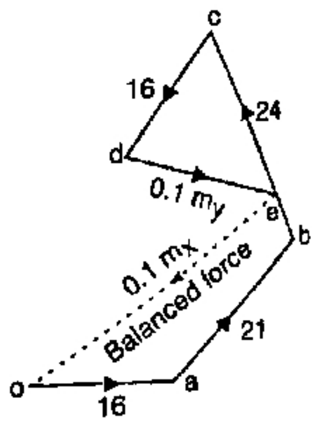


All dimensions in mm.

(*b*) Angular position of masses.



(*c*) Couple polygon.



(*d*) Force polygon.

Fig. 21.8

The angular position of the mass m_Y is obtained by drawing Om_Y in Fig. 21.8 (*b*), parallel to vector $d'o'$. By measurement, the angular position of m_Y is $\theta_Y = 12°$ in the clockwise direction from mass m_A (*i.e.* 200 kg). **Ans.**

2. Now draw the force polygon from the data given in Table 21.2 (column 4) as shown in Fig. 21.8 (*d*). The vector *eo* represents the balanced force. Since the balanced force is proportional to 0.1 m_X, therefore by measurement,

$$0.1\, m_X = \text{vector } eo = 35.5 \text{ kg-m} \quad \text{or} \quad m_X = 355 \text{ kg } \textbf{Ans.}$$

The angular position of the mass m_X is obtained by drawing Om_X in Fig. 21.8 (*b*), parallel to vector *eo*. By measurement, the angular position of m_X is $\theta_X = 145°$ in the clockwise direction from mass m_A (*i.e.* 200 kg). **Ans.**

Example 21.3. *Four masses A, B, C and D as shown below are to be completely balanced.*

	A	*B*	*C*	*D*
Mass (kg)	—	*30*	*50*	*40*
Radius (mm)	*180*	*240*	*120*	*150*

The planes containing masses B and C are 300 mm apart. The angle between planes containing B and C is 90°. B and C make angles of 210° and 120° respectively with D in the same sense. Find :

1. The magnitude and the angular position of mass A ; and

2. The position of planes A and D.

Solution. Given : r_A = 180 mm = 0.18 m ; m_B = 30 kg ; r_B = 240 mm = 0.24 m ; m_C = 50 kg ; r_C = 120 mm = 0.12 m ; m_D = 40 kg ; r_D = 150 mm = 0.15 m ; $\angle BOC = 90°$; $\angle BOD = 210°$; $\angle COD = 120°$

1. The magnitude and the angular position of mass A

Let m_A = Magnitude of Mass *A*,

x = Distance between the planes *B* and *D*, and

y = Distance between the planes *A* and *B*.

The position of the planes and the angular position of the masses is shown in Fig. 21.9 (*a*) and (*b*) respectively.

Assuming the plane *B* as the reference plane (*R.P.*) and the mass *B* (m_B) along the horizontal line as shown in Fig. 21.9 (*b*), the data may be tabulated as below :

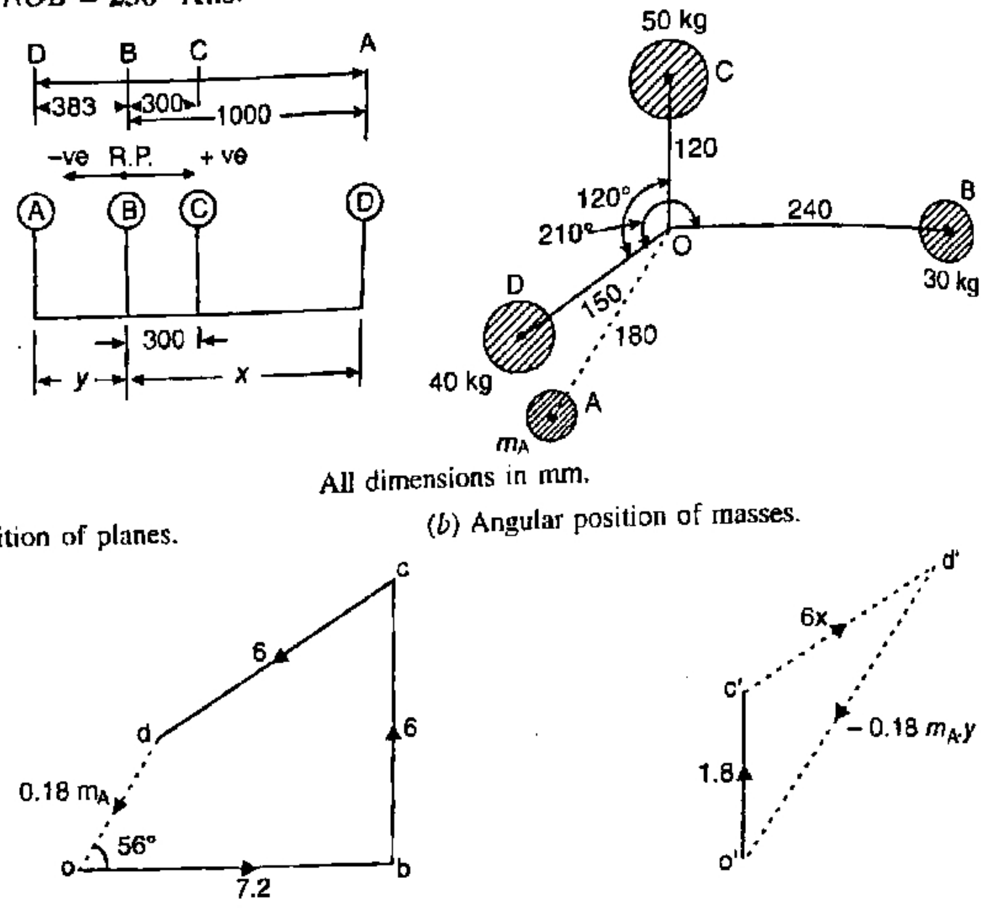
Table 21.3

Plane (1)	*Mass (m) kg* (2)	*Radius (r) m* (3)	*Cent.force* $\div \omega^2$ *(m.r) kg-m* (4)	*Distance from plane B (l) m* (5)	*Couple* $\div \omega^2$ *(m.r.l) kg-m*^{2} (6)
A	m_A	0.18	0.08 m_A	$-y$	$-0.18\, m_A\, y$
B (R.P)	30	0.24	7.2	0	0
C	50	0.12	6	0.3	1.8
D	40	0.15	6	x	$6x$

The magnitude and angular position of mass *A* may be determined by drawing the force polygon from the data given in Table 21.3 (Column 4), as shown in Fig. 21.9 (*c*), to some suitable scale. Since the masses are to be completely balanced, therefore the force polygon must be a closed figure. The closing side (*i.e.* vector *do*) is proportional to 0.18 m_A. By measurement,

$$0.18\ m_A = \text{Vector } do = 3.6 \text{ kg-m} \quad \text{or} \quad m_A = 20 \text{ kg} \textbf{ Ans.}$$

In order to find the angular position of mass *A*, draw *OA* in Fig. 21.9 (*b*) parallel to vector *do*. By measurement, we find that the angular position of mass *A* from mass *B* in the anticlockwise direction is $\angle AOB = 236°$ **Ans.**



All dimensions in mm.

(*a*) Position of planes. (*b*) Angular position of masses.

(*c*) Force polygon. (*d*) Couple polygon.

Fig. 21.9.

2. *Position of planes A and D*

The position of planes *A* and *D* may be obtained by drawing the couple polygon, as shown in Fig. 21.9 (*d*), from the data given in Table 21.3 (column 6). The couple polygon is drawn as discussed below :

1. Draw vector $o'c'$ parallel to *OC* and equal to 1.8 kg-m^2, to some suitable scale.

2. From points c' and o', draw lines parallel to *OD* and *OA* respectively, such that they intersect at point d'. By measurement, we find that

$$6\,x = \text{vector } c'd' = 2.3 \text{ kg-m}^2 \text{ or } x = 0.383 \text{ m}$$

We see from the couple polygon that the direction of vector $c'd'$ is opposite to the direction of mass *D*. Therefore the plane of mass *D* is 0.383 m or 383 mm towards left of plane *B* and not towards right of plane *B* as already assumed. **Ans.**

Again by measurement from couple polygon,

$$-0.18\,m_A.y = \text{vector } o'd' = 3.6 \text{ kg-m}^2$$

$$-0.18 \times 20\,y = 3.6 \quad \text{or} \quad y = -1 \text{ m}$$

The negative sign indicates that the plane *A* is not towards left of *B* as assumed but it is 1 m or 1000 mm towards right of plane *B*. **Ans.**

Example 21.4. *A, B, C and D are four masses carried by a rotating shaft at radii 100, 125, 200 and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg, and 4 kg respectively.*

Find the required mass A and the relative angular settings of the four masses so that the shaft shall be in complete balance.

Solution. Given : r_A = 100 mm = 0.1 m ; r_B = 125 mm = 0.125 m ; r_C = 200 mm = 0.2 m ; r_D = 150 mm = 0.15 m ; m_B = 10 kg ; m_C = 5 kg ; m_D = 4 kg

The position of planes is shown in Fig. 21.10 (*a*). Assuming the plane of mass *A* as the reference plane (*R.P.*), the data may be tabulated as below :

Table 21.4

Plane (1)	*Mass (m)* kg (2)	*Radius (r)* m (3)	*Cent. Force* ÷ ω^2 *(m.r)kg-m* (4)	*Distance from plane A (l)m* (5)	*Couple* ÷ ω^2 *(m.r.l) kg-m*^{2} (6)
A(R.P.)	m_A	0.1	0.1 m_A	0	0
B	10	0.125	1.25	0.6	0.75
C	5	0.2	1	1.2	1.2
D	4	0.15	0.6	1.8	1.08

First of all, the angular setting of masses *C* and *D* is obtained by drawing the couple polygon from the data given in Table 21.4 (column 6). Assume the position of mass *B* in the horizontal direction *OB* as shown in Fig. 21.10 (*b*). Now the couple polygon as shown in Fig. 21.10 (*c*) is drawn as discussed below :

1. Draw vector *o′ b′* in the horizontal direction (*i.e.* parallel to *OB*) and equal to 0.75 kg-m^2, to some suitable scale.
2. From points *o′* and *b′*, draw vectors *o′ c′* and *b′ c′* equal to 1.2 kg-m^2 and 1.08 kg-m^2 respectively. These vectors intersect at *c′*.
3. Now in Fig. 21.10 (*b*), draw *OC* parallel to vector *o′ c′* and *OD* parallel to vector *b′ c′*.

By measurement, we find that the angular setting of mass *C* from mass *B* in the anticlockwise direction, *i.e.*

$$\angle BOC = 240° \textbf{ Ans.}$$

and angular setting of mass *D* from mass *B* in the anticlockwise direction, *i.e.*

$$\angle BOD = 100° \textbf{ Ans.}$$

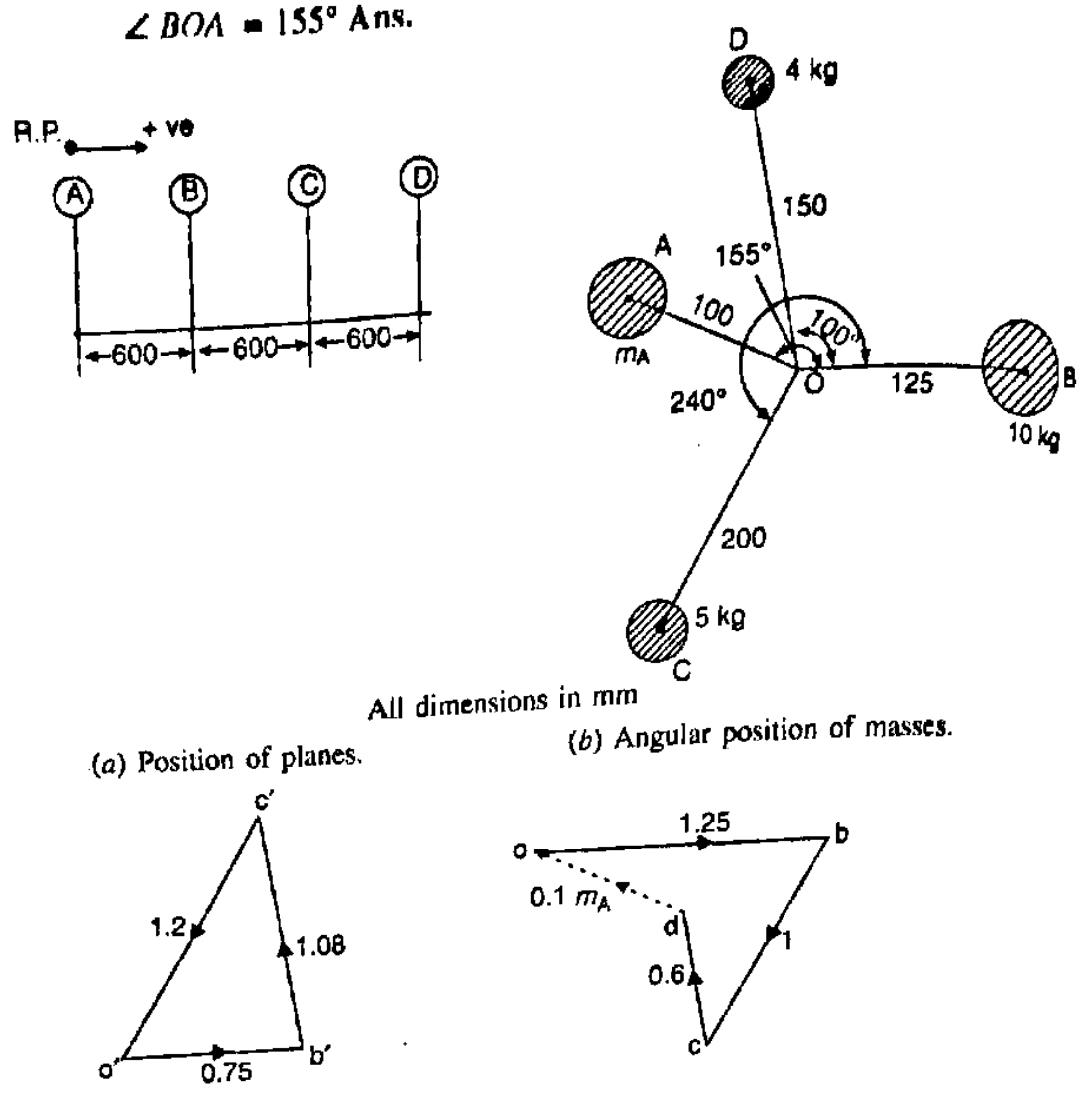
In order to find the required mass *A* (m_A) and its angular setting, draw the force polygon to some suitable scale, as shown in Fig. 21.10 (*d*), from the data given in Table 21.4 (column 4).

Since the closing side of the force polygon (vector *do*) is proportional to 0.1 m_A, therefore by measurement,

$$0.1\,m_A = 0.7 \text{ kg-m}^2 \quad \text{or} \quad m_A = 7 \text{ kg } \textbf{Ans.}$$

Now draw *OA* in Fig. 21.10 (*b*), parallel to vector *do*. By measurement, we find that the angular setting of mass *A* from mass *B* in the anticlockwise direction, *i.e.*

$$\angle BOA = 155° \textbf{ Ans.}$$



(*a*) Position of planes. (*b*) Angular position of masses.

(*c*) Couple polygon. (*d*) Force polygon.

Fig. 21.10

Example 21.5. *A shaft carries four masses in parallel planes A, B, C and D in this order along its length. The masses at B and C are 18 kg and 12.5 kg respectively, and each has an eccentricity of 60 mm. The masses at A and D have an eccentricity of 80 mm. The angle between the masses at B and C is 100° and that between the masses at B and A is 190°, both being measured in the same direction. The axial distance between the planes A and B is 100 mm and that between B and C is 200 mm. If the shaft is in complete dynamic balance, determine :*

1. The magnitude of the masses at A and D ; 2. the distance between planes A and D ; and 3. the angular position of the mass at D.

Solution. Given : $m_B = 18$ kg ; $m_C = 12.5$ kg ; $r_B = r_C = 60$ mm $= 0.06$ m ; $r_A = r_D = 80$ mm $= 0.08$ m ; $\angle BOC = 100°$; $\angle BOA = 190°$

1. *Magnitude of the masses at A and D*

Let M_A = Mass at *A*,

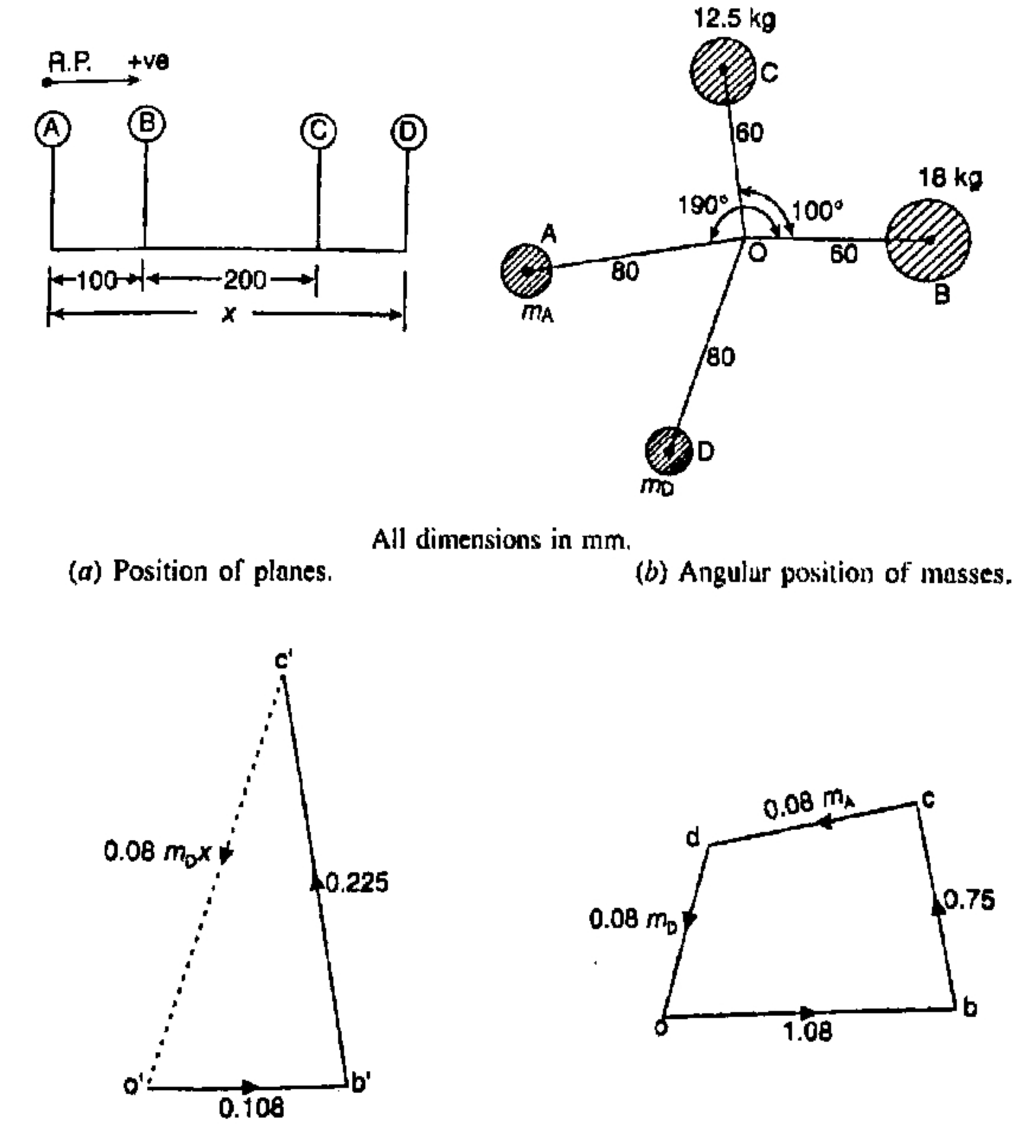
M_D = Mass at *D*, and

x = Distance between planes *A* and *D*.

The position of the planes and angular position of the masses is shown in Fig. 21.11 (*a*) and (*b*) respectively. The position of mass *B* is assumed in the horizontal direction, *i.e.* along *OB*. Taking the plane of mass *A* as the reference plane, the data may be tabulated as below :

Table 21.5

Plane (1)	*Mass* (*m*) *kg* (2)	*Eccentricity* (*r*) *m* (3)	*Cent. force* ÷ ω^2 (*m.r*) *kg-m* (4)	*Distance from plane A*(*l*)*m* (5)	*Couple* ÷ ω^2 (*m.r.l*) *kg-m*^{2} (6)
A (*R.P.*)	m_A	0.08	0.08 m_A	0	0
B	18	0.06	1.08	0.1	0.108
C	12.5	0.06	0.75	0.3	0.225
D	m_D	0.08	0.08 m_D	*x*	0.08 m_D . *x*



All dimensions in mm.

(*a*) Position of planes. (*b*) Angular position of masses.

(c) Couple polygon. (*d*) Force polygon.

Fig. 21.11

First of all, the direction of mass *D* is fixed by drawing the couple polygon to some suitable scale, as shown in Fig. 21.11 (*c*), from the data given in Table 21.5 (column 6). The closing side of the couple polygon (vector $c'o'$) is proportional to 0.08 $m_D.x$. By measurement, we find that

$$0.08\ m_D.x = \text{vector } c'o' = 0.235 \text{ kg-m}^2$$

In Fig. 21.11 (*b*), draw *OD* parallel to vector $c'o'$ to fix the direction of mass *D*.

Now draw the force polygon, to some suitable scale, as shown in Fig. 21.11 (*d*), from the data given in Table 21.5 (column 4), as discussed below :

1. Draw vector *ob* parallel to *OB* and equal to 1.08 kg-m.
2. From point *b*, draw vector *bc* parallel to *OC* and equal to 0.75 kg-m.
3. For the shaft to be in complete dynamic balance, the force polygon must be a closed figure. Therefore from point *c*, draw vector *cd* parallel to *OA* and from point *o* draw vector *od* parallel to *OD*. The vectors *cd* and *od* intersect at *d*. Since the vector *cd* is proportional to 0.08 m_A, therefore by measurement

$$0.08\ m_A = \text{vector } cd = 0.77 \text{ kg-m} \quad \text{or} \quad m_A = 9.625 \text{ kg } \textbf{Ans.}$$

and vector *do* is proportional to 0.08 m_D, therefore by measurement,

$$0.08\ m_D = \text{vector } do = 0.65 \text{ kg-m} \quad \text{or} \quad m_D = 8.125 \text{ kg } \textbf{Ans.}$$

2. ***Distance between planes A and D***

From equation (*i*),

$$0.08\ m_D.x = 0.235 \text{ kg-m}^2$$

$$0.08 \times 8.125 \times x = 0.235 \text{ kg-m}^2 \quad \text{or} \quad 0.65\ x = 0.235$$

$$\therefore \quad x = \frac{0.235}{0.65} = 0.3615 \text{m} = 361.5 \text{ mm } \textbf{Ans.}$$

3. ***Angular position of mass at D***

By measurement from Fig. 21.11 (*b*), we find that the angular position of mass at *D* from mass *B* in the anticlockwise direction, *i.e.* $\angle BOD = 251°$ **Ans.**

22

Balancing of Reciprocating Masses

Features

1. *Introduction.*
2. *Primary and Secondary Unbalanced Forces of Reciprocating Masses.*
3. *Partial Balancing of Unbalanced Primary Force in a Reciprocating Engine.*
4. *Partial Balancing of Locomotives.*
5. *Effect of Partial Balancing of Reciprocating Parts of Two Cylinder Locomotives.*
6. *Variation of Tractive Force.*
7. *Swaying Couple.*
8. *Hammer Blow.*
9. *Balancing of Coupled Locomotives.*
10. *Balancing of Primary Forces of Multi-cylinder In-line Engines.*
11. *Balancing of Secondary Forces of Multi-cylinder In-line Engines.*
12. *Balancing of Radial Engines (Direct and Reverse Crank Method).*
13. *Balancing of V-engines.*

22.1. Introduction

We have discussed in Chapter 15 (Art. 15.10), the various forces acting on the reciprocating parts of an engine. The resultant of all the forces acting on the body of the engine due to inertia forces only is known as ***unbalanced force*** or ***shaking force***. Thus if the resultant of all the forces due to inertia effects is zero, then there will be no unbalanced force, but even then an unbalanced couple or shaking couple will be present.

Consider a horizontal reciprocating engine mechanism as shown in Fig. 22.1.

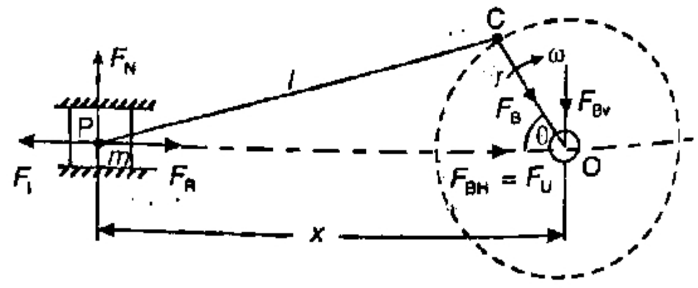


Fig. 22.1. Reciprocating engine mechanism.

Let F_R = Force required to accelerate the reciprocating parts,

F_I = Inertia force due to reciprocating parts,

F_N = Force on the sides of the cylinder walls or normal force acting on the cross-head guides, and

F_B = Force acting on the crankshaft bearing or main bearing.

Since F_R and F_I are equal in magnitude but opposite in direction, therefore they balance each other. The horizontal component of F_B (*i.e.* F_{BH}) acting along the line of reciprocation is also equal and opposite to F_I. This force $F_{BH} = F_U$ is an unbalanced force or shaking force and required to be properly balanced.

The force on the sides of the cylinder walls (F_N) and the vertical component of F_B (*i.e.* F_{BV}) are equal and opposite and thus form a shaking couple of magnitude $F_N \times x$ or $F_{BV} \times x$.

From above we see that the effect of the reciprocating parts is to produce a shaking force and a shaking couple. Since the shaking force and a shaking couple vary in magnitude and direction during the engine cycle, therefore they cause very objectionable vibrations.

Thus the purpose of balancing the reciprocating masses is to eliminate the shaking force and a shaking couple. In most of the mechanisms, we can reduce the shaking force and a shaking couple by adding appropriate balancing mass, but it is usually not practical to eliminate them completely. In other words, the reciprocating masses are only partially balanced.

Note : The masses rotating with the crankshaft are normally balanced and they do not transmit any unbalanced or shaking force on the body of the engine.

22.2. Primary and Secondary Unbalanced Forces of Reciprocating Masses

Consider a reciprocating engine mechanism as shown in Fig. 22.1.

Let m = Mass of the reciprocating parts,

l = Length of the connecting rod PC,

r = Radius of the crank OC,

θ = Angle of inclination of the crank with the line of stroke PO,

ω = Angular speed of the crank,

n = Ratio of length of the connecting rod to the crank radius = l / r.

We have already discussed in Art. 15.8 that the acceleration of the reciprocating parts is approximately given by the expression,

$$a_R = \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

∴ Inertia force due to reciprocating parts or force required to accelerate the reciprocating parts,

$$F_I = F_R = \text{Mass} \times \text{acceleration} = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

We have discussed in the previous article that the horizontal component of the force exerted on the crank shaft bearing (*i.e.* F_{BH}) is equal and opposite to inertia force (F_I). This force is an unbalanced one and is denoted by F_U.

∴ Unbalanced force,

$$F_U = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = m.\omega^2 \cdot r\cos\theta + m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n} = F_P + F_S$$

The expression $(m \cdot \omega^2 \cdot r\cos\theta)$ is known as ***primary unbalanced force*** and $\left(m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n}\right)$ is called ***secondary unbalanced force***.

$\therefore$ Primary unbalanced force, $F_P = m \cdot \omega^2 \cdot r \cos\theta$

and secondary unbalanced force, $F_S = m \cdot \omega^2 \cdot r \times \dfrac{\cos 2\theta}{n}$

Notes: 1. The primary unbalanced force is maximum, when $\theta = 0°$ or $180°$. Thus, the primary force is maximum twice in one revolution of the crank. The maximum primary unbalanced force is given by

$$F_{P(max)} = m \cdot \omega^2 \cdot r$$

2. The secondary unbalanced force is maximum, when $\theta = 0°, 90°, 180°$ and $360°$. Thus, the secondary force is maximum four times in one revolution of the crank. The maximum secondary unbalanced force is given by

$$F_{S(max)} = m \cdot \omega^2 \times \frac{r}{n}$$

3. From above we see that maximum secondary unbalanced force is $1/n$ times the maximum primary unbalanced force.

4. In case of moderate speeds, the secondary unbalanced force is so small that it may be neglected as compared to primary unbalanced force.

5. The unbalanced force due to reciprocating masses varies in magnitude but constant in direction while due to the revolving masses, the unbalanced force is constant in magnitude but varies in direction.

22.3. Partial Balancing of Unbalanced Primary Force in a Reciprocating Engine

The primary unbalanced force $(m \cdot \omega^2 \cdot r \cos\theta)$ may be considered as the component of the centrifugal force produced by a rotating mass m placed at the crank radius r, as shown in Fig. 22.2.

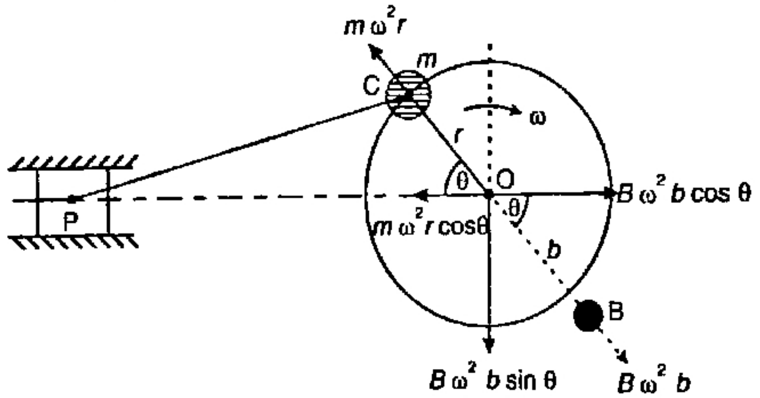


Fig. 22.2. Partial balancing of unbalanced primary force in a reciprocating engine.

The primary force acts from O to P along the line of stroke. Hence, balancing of primary force is considered as equivalent to the balancing of mass m rotating at the crank radius r. This is balanced by having a mass B at a radius b, placed diametrically opposite to the crank pin C.

We know that centrifugal force due to mass B,

$$= B \cdot \omega^2 \cdot b$$

and horizontal component of this force acting in opposite direction of primary force

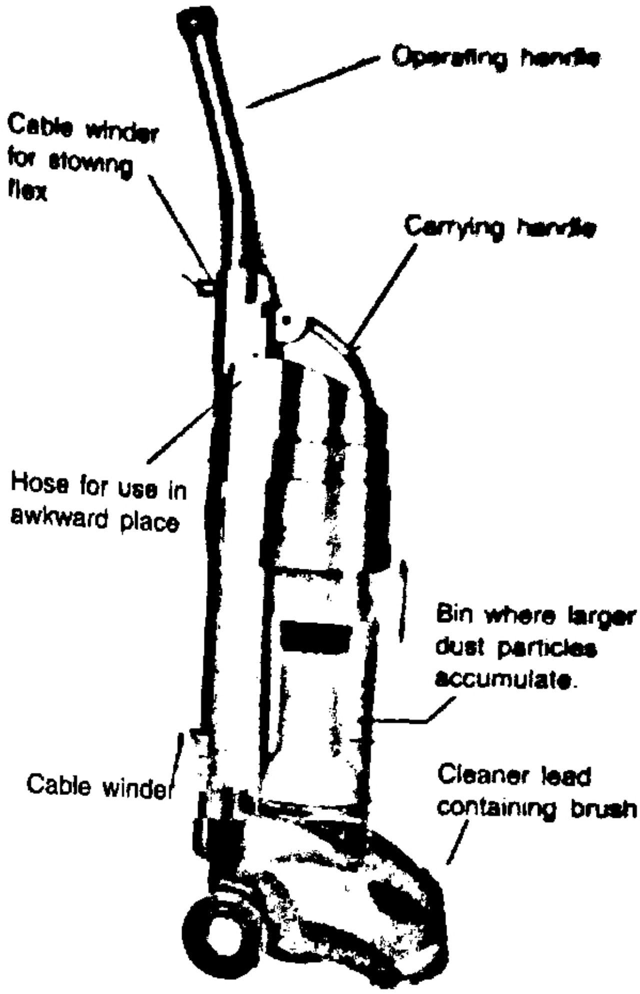
$$= B \cdot \omega^2 \cdot b \cos\theta$$

The primary force is balanced, if

$$B \cdot \omega^2 \cdot b \cos\theta = m \cdot \omega^2 \cdot r \cos\theta \quad \text{or} \quad B.b = m.r$$

A little consideration will show, that the primary force is completely balanced if $B.b = m.r$, but the centrifugal force produced due to the revolving mass B has also a vertical component (perpendicular to the line of stroke) of magnitude $B \cdot \omega^2 \cdot b \sin\theta$. This force remains unbalanced. The maximum value of this force is equal to $B \cdot \omega^2 \cdot b$ when θ is 90° and 270°, which is same as the maximum value of the primary force $m \cdot \omega^2 \cdot r$.

From the above discussion, we see that in the first case, the primary unbalanced force acts along the line of stroke whereas in the second case, the unbalanced force acts along the perpendicular to the line of stroke. The maximum value of the force remains same in both the cases. It is thus obvious, that the effect of the above method of balancing is to change the direction of the maximum unbalanced force from the line of stroke to the perpendicular of line of stroke. As a compromise let a fraction 'c' of the reciprocating masses is balanced, such that



Cyclone cleaner.

$$c.m.r = B.b$$

∴ Unbalanced force along the line of stroke

$$= m \cdot \omega^2 \cdot r\cos\theta - B \cdot \omega^2 \cdot b\cos\theta$$

$$= m \cdot \omega^2 \cdot r\cos\theta - c \cdot m \cdot \omega^2 \cdot r\cos\theta \qquad \ldots (\because\ B.b = c.m.r)$$

$$= (1-c)m \cdot \omega^2 \cdot r\cos\theta$$

and unbalanced force along the perpendicular to the line of stroke

$$= B \cdot \omega^2 \cdot b\sin\theta = c \cdot m \cdot \omega^2 \cdot r\sin\theta$$

∴ Resultant unbalanced force at any instant

$$= \sqrt{\left[(1-c)m \cdot \omega^2 \cdot r\cos\theta\right]^2 + \left[c \cdot m \cdot \omega^2 \cdot r\sin\theta\right]^2}$$

$$= m \cdot \omega^2 \cdot r\sqrt{(1-c)^2\cos^2\theta + c^2\sin^2\theta}$$

Note : If the balancing mass is required to balance the revolving masses as well as reciprocating masses, then

$$B.b = m_1 \cdot r + c \cdot m \cdot r = (m_1 + c \cdot m)r$$

where m_1 = Magnitude of the revolving masses, and

m = magnitude of the reciprocating masses.

Example 22.1. *A single cylinder reciprocating engine has speed 240 r.p.m., stroke 300 mm, mass of reciprocating parts 50 kg, mass of revolving parts at 150 mm radius 37 kg. If two third of the reciprocating parts and all the revolving parts are to be balanced, find : 1. The balance mass required at a radius of 400 mm, and 2. The residual unbalanced force when the crank has rotated 60° from inner dead centre.*

Solution. Given : N = 240 r.p.m. or $\omega = 2\pi \times 240/60$ = 25.14 rad/s ; Stroke = 300 mm = 0.3 m; m = 50 kg ; m_1 = 37 kg ; r = 150 mm = 0.15 m ; c = 2/3

1. ***Balance mass required***

Let B = Balance mass required, and

b = Radius of rotation of the balance mass = 400 mm = 0.4 m ... (Given)

We know that

$$B.b = (m_1 + c.m)\,r$$

$$B \times 0.4 = \left(37 + \frac{2}{3} \times 50\right) 0.15 = 10.55 \quad \text{or} \quad B = 26.38 \text{ kg } \textbf{Ans.}$$

2. ***Residual unbalanced force***

Let θ = Crank angle from inner dead centre = 60° ... (Given)

We know that residual unbalanced force

$$= m \cdot \omega^2 \cdot r\sqrt{(1-c)^2 \cos^2\theta + c^2 \sin^2\theta}$$

$$= 50(25.14)^2 0.15\sqrt{\left(1 - \frac{2}{3}\right)^2 \cos^2 60° + \left(\frac{2}{3}\right)^2 \sin^2 60°} \text{ N}$$

$$= 4740 \times 0.601 = 2849 \text{ N } \textbf{Ans.}$$

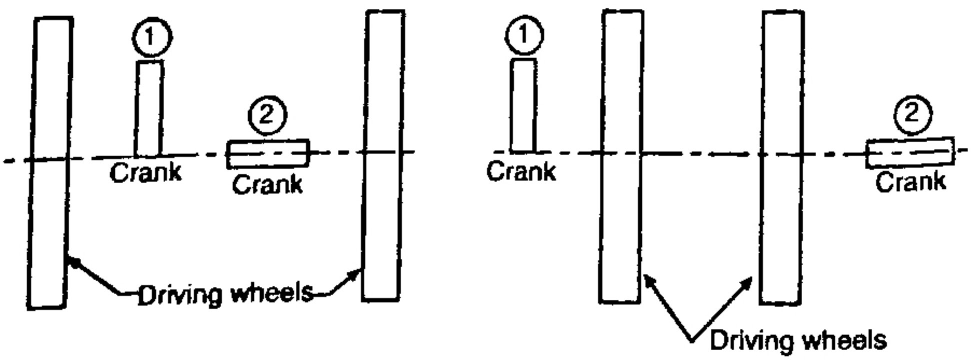
22.4. Partial Balancing of Locomotives

The locomotives, usually, have two cylinders with cranks placed at right angles to each other in order to have uniformity in turning moment diagram. The two cylinder locomotives may be classified as :

1. Inside cylinder locomotives ; and **2.** Outside cylinder locomotives.

In the ***inside cylinder locomotives***, the two cylinders are placed in between the planes of two driving wheels as shown in Fig. 22.3 (*a*) ; whereas in the ***outside cylinder locomotives***, the two cylinders are placed outside the driving wheels, one on each side of the driving wheel, as shown in Fig. 22.3 (*b*). The locomotives may be

(*a*) Single or uncoupled locomotives ; and (*b*) Coupled locomotives.



(*a*) Inside cylinder locomotives. (*b*) Outside cylinder locomotives.

Fig. 22.3

A single or *uncoupled locomotive* is one, in which the effort is transmitted to one pair of wheels only ; whereas in *coupled locomotives*, the driving wheels are connected to the leading and trailing wheel by an outside coupling rod.

22.5. Effect of Partial Balancing of Reciprocating Parts of Two Cylinder Locomotives

We have discussed in the previous article that the reciprocating parts are only partially balanced. Due to this partial balancing of the reciprocating parts, there is an unbalanced primary force along the line of stroke and also an unbalanced primary force perpendicular to the line of stroke. The effect of an unbalanced primary force along the line of stroke is to produce.

1. Variation in tractive force along the line of stroke ; and 2. Swaying couple.

The effect of an unbalanced primary force perpendicular to the line of stroke is to produce variation in pressure on the rails, which results in hammering action on the rails. The maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as a *hammer blow*. We shall now discuss the effects of an unbalanced primary force in the following articles.

22.6. Variation of Tractive Force

The resultant unbalanced force due to the two cylinders, along the line of stroke, is known as *tractive force*. Let the crank for the first cylinder be inclined at an angle θ with the line of stroke, as shown in Fig. 22.4. Since the crank for the second cylinder is at right angle to the first crank, therefore the angle of inclination for the second crank will be $(90° + \theta)$.

Let m = Mass of the reciprocating parts per cylinder, and

c = Fraction of the reciprocating parts to be balanced.

We know that unbalanced force along the line of stroke for cylinder 1

$$= (1-c)m.\omega^2.r\cos\theta$$

Similarly, unbalanced force along the line of stroke for cylinder 2,

$$= (1-c)m.\omega^2 \cdot r\cos(90°+\theta)$$

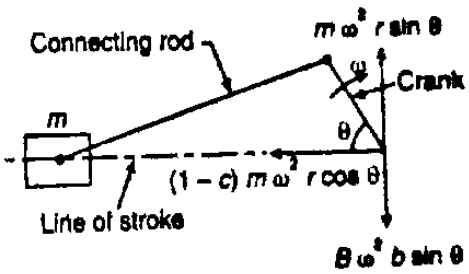


Fig. 22.4. Variation of tractive force.

∴ As per definition, the tractive force,

F_T = Resultant unbalanced force along the line of stroke

$$= (1-c)m.\omega^2.r\cos\theta + (1-c)m.\omega^2.r\cos(90°+\theta)$$

$$= (1-c)m.\omega^2.r(\cos\theta - \sin\theta)$$

The tractive force is maximum or minimum when $(\cos\theta - \sin\theta)$ is maximum or minimum. For $(\cos\theta - \sin\theta)$ to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta - \sin\theta) = 0 \quad \text{or} \quad -\sin\theta - \cos\theta = 0 \quad \text{or} \quad -\sin\theta = \cos\theta$$

$$\therefore \quad \tan\theta = -1 \quad \text{or} \quad \theta = 135° \quad \text{or} \quad 315°$$

Thus, the tractive force is maximum or minimum when θ = 135° or 315°.

∴ Maximum and minimum value of the tractive force or the variation in tractive force

$$= \pm(1-c)\,m.\omega^2.r(\cos 135° - \sin 135°) = \pm\sqrt{2}\,(1-c)\,m.\omega^2.r$$

22.7. Swaying Couple

The unbalanced forces along the line of stroke for the two cylinders constitute a couple about the centre line YY between the cylinders as shown in Fig. 22.5.

This couple has swaying effect about a vertical axis, and tends to sway the engine alternately in clockwise and anticlockwise directions. Hence the couple is known as ***swaying couple.***

Let a = Distance between the centre lines of the two cylinders.

∴ Swaying couple

$$= (1-c)m.\omega^2.r\cos\theta\times\frac{a}{2}$$

$$-(1-c)m.\omega^2.r\cos(90°+\theta)\frac{a}{2}$$

$$= (1-c)m.\omega^2.r\times\frac{a}{2}(\cos\theta+\sin\theta)$$

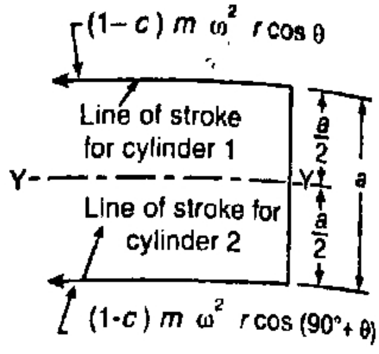


Fig. 22.5. Swaying couple.

The swaying couple is maximum or minimum when $(\cos\theta+\sin\theta)$ is maximum or minimum. For $(\cos\theta+\sin\theta)$ to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta+\sin\theta)=0 \quad \text{or} \quad -\sin\theta+\cos\theta=0 \quad \text{or} \quad -\sin\theta=-\cos\theta$$

$$\therefore \quad \tan\theta=1 \quad \text{or} \quad \theta=45° \quad \text{or} \quad 225°$$

Thus, the swaying couple is maximum or minimum when θ = 45° or 225°.

∴ Maximum and minimum value of the swaying couple

$$= \pm(1-c)m.\omega^2.r\times\frac{a}{2}(\cos 45°+\sin 45°)=\pm\frac{a}{\sqrt{2}}(1-c)m.\omega^2.r$$

Note : In order to reduce the magnitude of the swaying couple, revolving balancing masses are introduced. But, as discussed in the previous article, the revolving balancing masses cause unbalanced forces to act at right angles to the line of stroke. These forces vary the downward pressure of the wheels on the rails and cause oscillation of the locomotive in a vertical plane about a horizontal axis. Since a swaying couple is more harmful than an oscillating couple, therefore a value of 'c' from 2/3 to 3/4, in two-cylinder locomotives with two pairs of coupled wheels, is usually used. But in large four cylinder locomotives with three or more pairs of coupled wheels, the value of 'c' is taken as 2/5.

22.8. Hammer Blow

We have already discussed that the maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as ***hammer blow.***

We know that the unbalanced force along the perpendicular to the line of stroke due to the balancing mass B, at a radius b, in order to balance reciprocating parts only is $B.\omega^2.b\sin\theta$. This force will be maximum when $\sin\theta$ is unity, *i.e.* when θ = 90° or 270°.

∴ Hammer blow = $B.\omega^2.b$ (Substituiting $\sin\theta = 1$)

The effect of hammer blow is to cause the variation in pressure between the wheel and the rail. This variation is shown in Fig. 22.6, for one revolution of the wheel.

Let P be the downward pressure on the rails (or static wheel load).

∴ Net pressure between the wheel and the rail

$$= P \pm B.\omega^2.b$$

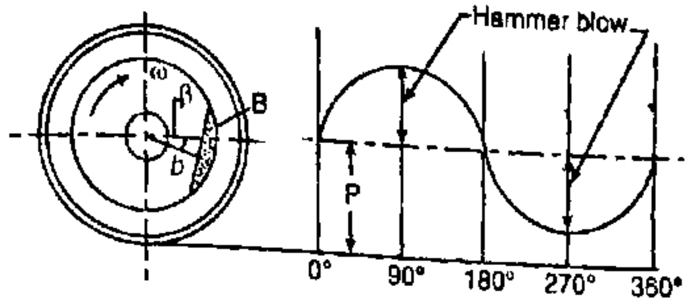


Fig. 22.6. Hammer blow.

If $(P - B.\omega^2.b)$ is *negative*, then the wheel will be lifted from the rails. Therefore the limiting condition in order that the wheel does not lift from the rails is given by

$$P = B.\omega^2.b$$

and the permissible value of the angular speed,

$$\omega = \sqrt{\frac{P}{B.b}}$$

Example 22.2. *An inside cylinder locomotive has its cylinder centre lines 0.7 m apart and has a stroke of 0.6 m. The rotating masses per cylinder are equivalent to 150 kg at the crank pin, and the reciprocating masses per cylinder to 180 kg. The wheel centre lines are 1.5 m apart. The cranks are at right angles.*

The whole of the rotating and 2/3 of the reciprocating masses are to be balanced by masses placed at a radius of 0.6 m. Find the magnitude and direction of the balancing masses.

Find the fluctuation in rail pressure under one wheel, variation of tractive effort and the magnitude of swaying couple at a crank speed of 300 r.p.m.

Solution. Given : $a = 0.7$ m; $l_B = l_C = 0.6$ m or $r_B = r_C = 0.3$ m; $m_1 = 150$ kg; $m_2 = 180$ kg; $c = 2/3$; $r_A = r_D = 0.6$ m; $N = 300$ r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s

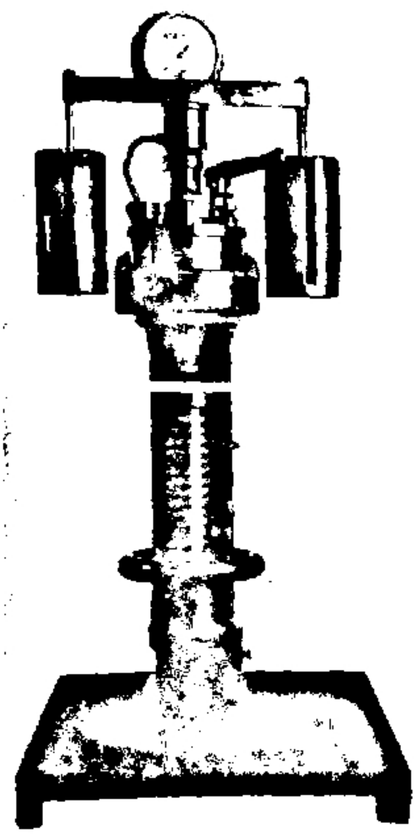
We know that the equivalent mass of the rotating parts to be balanced per cylinder at the crank pin,

$$m = m_B = m_C = m_1 + c.m_2 = 150 + \frac{2}{3} \times 180 = 270 \text{ kg}$$

Magnitude and direction of the balancing masses

Let m_A and m_D = Magnitude of the balancing masses

θ_A and θ_D = Angular position of the balancing masses m_A and m_D from the first crank B.



This Brinel hardness testing machine is used to test the hardness of the metal.

Note : This picture is given as additional information.

The magnitude and direction of the balancing masses may be determined graphically as discussed below :

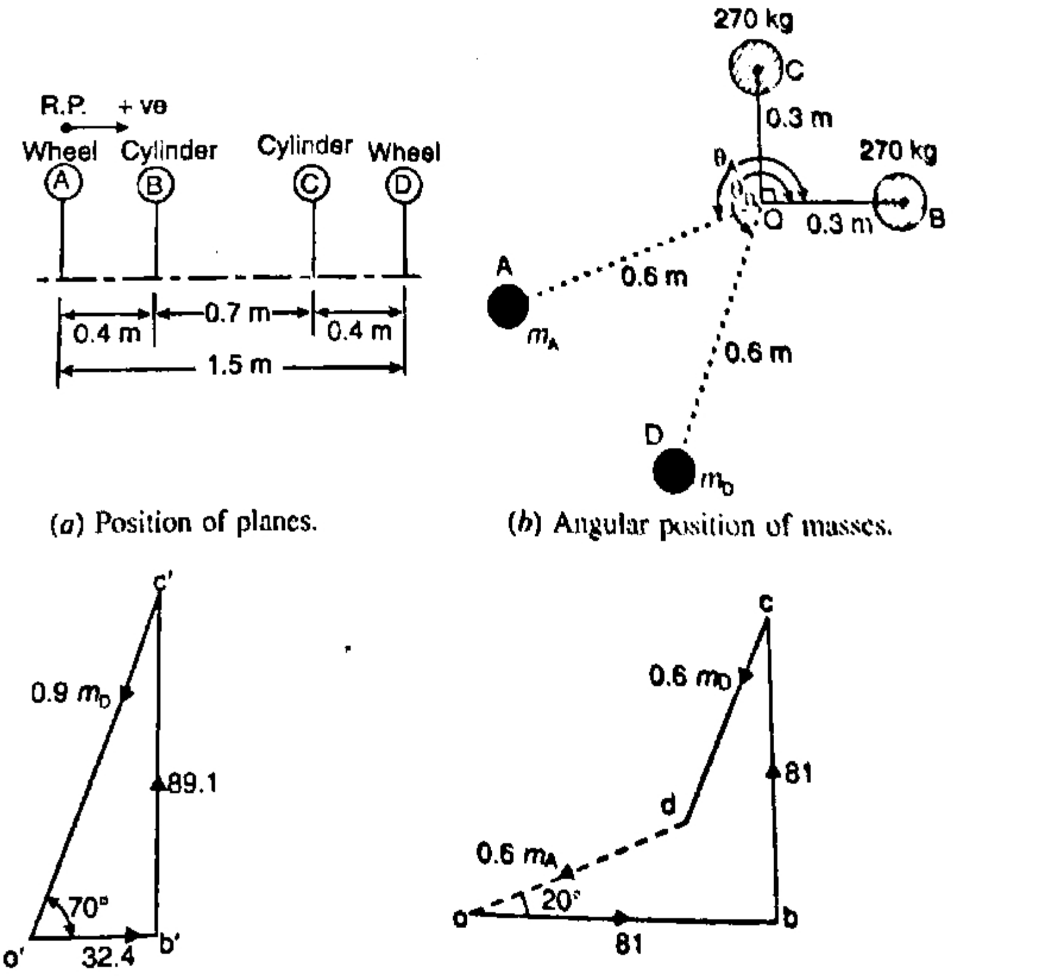
1. First of all, draw the space diagram to show the positions of the planes of the wheels and the cylinders, as shown in Fig. 22.7 (*a*). Since the cranks of the cylinders are at right angles, therefore assuming the position of crank of the cylinder *B* in the horizontal direction, draw *OC* and *OB* at right angles to each other as shown in Fig. 22.7 (*b*).
2. Tabulate the data as given in the following table. Assume the plane of wheel *A* as the reference plane.

Table 22.1

Plane (1)	*mass.* (*m*) *kg* (2)	*Radius* (*r*)*m* (3)	*Cent. force* ÷ ω^2 (*m.r*) *kg-m* (4)	*Distance from plane A* (*l*)*m* (5)	*Couple* ÷ ω^2 (*m.r.l*) *kg-m*² (6)
A (*R.P.*)	m_A	0.6	0.6 m_A	0	0
B	270	0.3	81	0.4	32.4
C	270	0.3	81	1.1	89.1
D	m_D	0.6	0.6m_D	1.5	0.9 m_D

3. Now, draw the couple polygon from the data given in Table 22.1 (column 6), to some suitable scale, as shown in Fig 22.7 (*c*). The closing side *c′o′* represents the balancing couple and it is proportional to 0.9 m_D. Therefore, by measurement,

$$0.9\ m_D = \text{vector } c'o' = 94.5 \text{ kg-m}^2 \quad \text{or} \quad m_D = 105 \text{ kg} \quad \text{Ans.}$$



(*a*) Position of planes. (*b*) Angular position of masses.

(*c*) Couple polygon. (*d*) Force polygon.

Fig. 22.7

4. To determine the angular position of the balancing mass D, draw OD in Fig. 22.7 (b) parallel to vector $c'o'$. By measurement,

$$\theta_D = 250° \text{ Ans.}$$

5. In order to find the balancing mass A, draw the force polygon from the data given in Table 22.1 (column 4), to some suitable scale, as shown in Fig. 22.7 (d). The vector do represents the balancing force and it is proportional to $0.6\, m_A$. Therefore by measurement,

$$0.6\, m_A = \text{vector } do = 63 \text{ kg-m or } m_A = 105 \text{ kg Ans.}$$

6. To determine the angular position of the balancing mass A, draw OA in Fig. 22.7 (b) parallel to vector *do*. By measurement,

$$\theta_A = 200° \text{ Ans.}$$

Fluctuation in rail pressure

We know that each balancing mass

$$= 105 \text{ kg}$$

∴ Balancing mass for rotating masses,

$$= \frac{m_1}{m} \times 105 = \frac{150}{270} \times 105 = 58.3 \text{ kg}$$

and balancing mass for reciprocating masses,

$$B = \frac{c.m_2}{m} \times 105 = \frac{2}{3} \times \frac{180}{270} \times 105 = 46.6 \text{ kg}$$

This balancing mass of 46.6 kg for reciprocating masses gives rise to the centrifugal force.

∴ Fluctuation in rail pressure or hammer blow

$$= B.\omega^2.b = 46.6\,(31.42)^2\,0.6 = 27\,602 \text{ N Ans.} \qquad \ldots (\because b = r_A = r_D)$$

Variation of tractive effort

We know that maximum variation of tractive effort

$$= \pm\sqrt{2}(1-c)m_2.\omega^2.r = \pm\sqrt{2}\left(1-\frac{2}{3}\right)180(31.42)^2\,0.3 \text{N}$$

$$= \pm\, 25\,127 \text{ N Ans.} \qquad \ldots (\because r = r_B = r_C)$$

Swaying couple

We know that maximum swaying couple

$$= \frac{a(1-c)}{\sqrt{2}} \times m_2.\omega^2.r = \frac{0.7\left(1-\frac{2}{3}\right)}{\sqrt{2}} \times 180(31.42)^2\,0.3 \text{ N-m}$$

$$= 8797 \text{ N-m Ans.}$$

Example 22.3 *The three cranks of a three cylinder locomotive are all on the same axle and are set at 120°. The pitch of the cylinders is 1 metre and the stroke of each piston is 0.6 m. The reciprocating masses are 300 kg for inside cylinder and 260 kg for each outside cylinder and the planes of rotation of the balance masses are 0.8 m from the inside crank.*

If 40% of the reciprocating parts are to be balanced, find :

1. the magnitude and the position of the balancing masses required at a radius of 0.6 m ; and

2. the hammer blow per wheel when the axle makes 6 r.p.s.

Solution. Given : $\angle AOB = \angle BOC = \angle COA = 120°$; $l_A = l_B = l_C = 0.6$ m or $r_A = r_B = r_C = 0.3$ m ; $m_I = 300$ kg ; $m_O = 260$ kg ; $c = 40\% = 0.4$; $b_1 = b_2 = 0.6$ m ; $N = 6$ r.p.s. $= 6 \times 2\pi = 37.7$ rad/s

Since 40% of the reciprocating masses are to be balanced, therefore mass of the reciprocating parts to be balanced for each outside cylinder,

$$m_A = m_C = c \times m_O = 0.4 \times 260 = 104 \text{ kg}$$

and mass of the reciprocating parts to be balanced for inside cylinder,

$$m_B = c \times m_I = 0.4 \times 300 = 120 \text{ kg}$$

1. *Magnitude and position of the balancing masses*

Let B_1 and B_2 = Magnitude of the balancing masses in kg,

θ_1 and θ_2 = Angular position of the balancing masses B_1 and B_2 from crank A.

The magnitude and position of the balancing masses may be determined graphically as discussed below :

1. First of all, draw the position of planes and cranks as shown in Fig. 22.8 (*a*) and (*b*) respectively. The position of crank A is assumed in the horizontal direction.
2. Tabulate the data as given in the following table. Assume the plane of balancing mass B_1 (*i.e.* plane 1) as the reference plane.

Table 22.2

Plane (*1*)	*Mass* (*m*)*kg* (*2*)	*Radius* (*r*) *m* (*3*)	*Cent. force* $\div \omega^2$ (*m.r*) *kg-m* (*4*)	*Distance from plane1* (*l*)*m* (*5*)	*Couple* $\div \omega^2$ (*m.r.l.*) *kg-m²* (*6*)
A	104	0.3	31.2	– 0.2	– 6.24
1 (*R.P.*)	B_1	0.6	0.6 B_1	0	0
B	120	0.3	36	0.8	28.8
2	B_2	0.6	0.6 B_2	1.6	0.96 B_2
C	104	0.3	31.2	1.8	56.16

3. Now draw the couple polygon with the data given in Table 22.2 (column 6), to some suitable scale, as shown in Fig. 22.8 (*c*). The closing side $c'o'$ represents the balancing couple and it is proportional to 0.96 B_2. Therefore, by measurement,

$$0.96\, B_2 = \text{vector } c'o' = 55.2 \text{ kg-m}^2 \text{ or } B_2 = 57.5 \text{ kg \textbf{Ans.}}$$

4. To determine the angular position of the balancing mass B_2, draw OB_2 parallel to vector $c'o'$ as shown in Fig. 22.8 (*b*). By measurement,

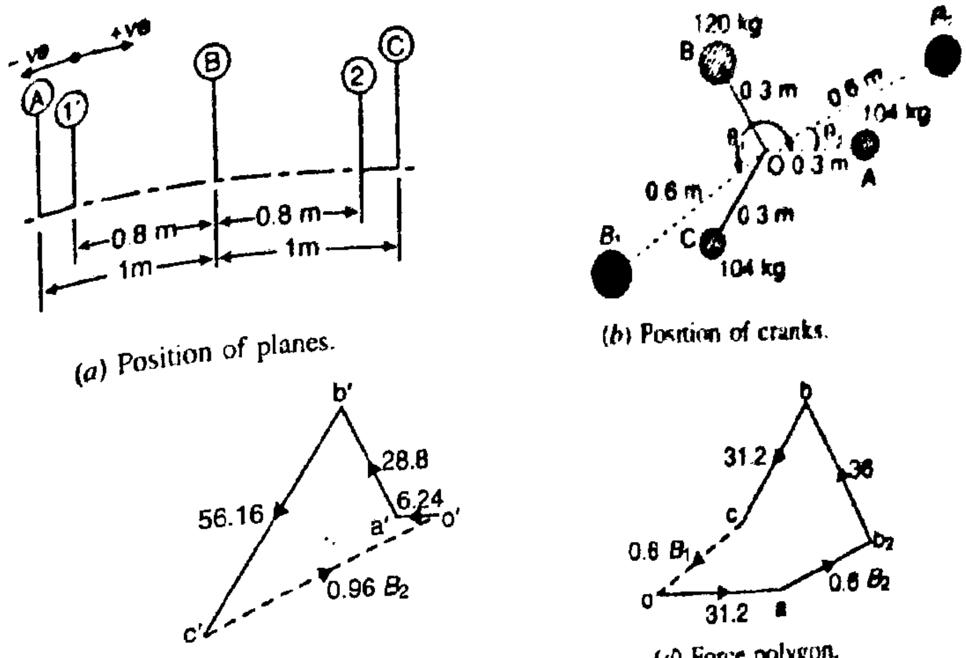
$$\theta_2 = 24° \text{ \textbf{Ans.}}$$

5. In order to find the balance mass B_1, draw the force polygon with the data given in Table 22.2 (column 4), to some suitable scale, as shown in Fig. 22.8 (*d*). The closing side *co* represents the balancing force and it is proportional to 0.6 B_1. Therefore, by measurement

$$0.6\, B_1 = \text{vector } co = 34.5 \text{ kg-m} \quad \text{or} \quad B_1 = 57.5 \text{ kg \textbf{Ans.}}$$

6. To determine the angular position of the balancing mass B_1, draw OB_1 parallel to vector *co*, as shown in Fig. 22.8 (*b*). By measurement,

$$\theta_1 = 215° \text{ \textbf{Ans.}}$$



(*a*) Position of planes.

(*b*) Position of cranks.

(*c*) Couple polygon.

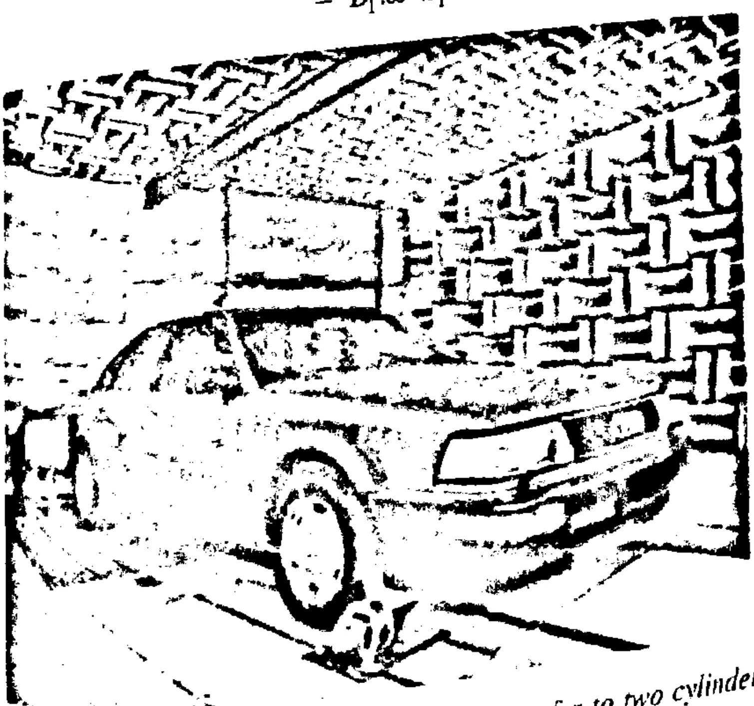
(*d*) Force polygon.

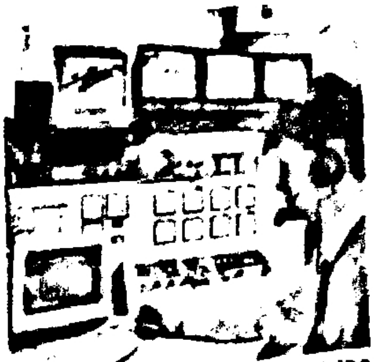
Fig. 22.8

2. Hammer blow per wheel

We know that hammer blow per wheel

$$= B_1.\omega^2.b_1 = 57.5\ (37.7)^2\ 20.6 = 49\ 035 \text{ N } \textbf{Ans.}$$





This chamber is used to test the acoustics of a vehicle so that the noise it produces can be reduced. The panels in the walls and ceiling of the room absorb the sound which is monitored (above)

Note : This picture is given as additional information.

Example 22.4. *The following data refer to two cylinder locomotive with cranks at 90° :*

Reciprocating mass per cylinder = 300 kg ; Crank radius = 0.3 m ; Driving wheel diameter = 1.8 m ; Distance between cylinder centre lines = 0.65 m ; Distance between the driving wheel central planes = 1.55 m.

Determine : 1. the fraction of the reciprocating masses to be balanced, if the hammer blow is not to exceed 46 kN at 96.5 km. p.h. ; 2. the variation in tractive effort ; and 3. the maximum swaying couple.

Solution. Given : $m = 300$ kg ; $r = 0.3$ m ; $D = 1.8$ m or $R = 0.9$ m ; $a = 0.65$ m ; Hammer blow $= 46$ kN $= 46 \times 10^3$ N ; $v = 96.5$ km/h $= 26.8$ m/s

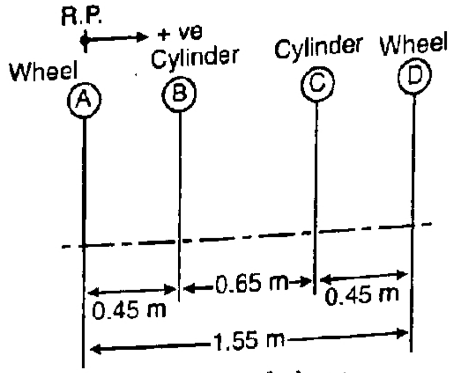
1. ***Fraction of the reciprocating masses to be balanced***

Let c = Fraction of the reciprocating masses to be balanced, and

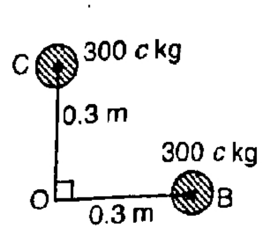
B = Magnitude of balancing mass placed at each of the driving wheels at radius b.

We know that the mass of the reciprocating parts to be balanced

$= c.m = 300c$ kg



(*a*) Position of planes.



(*b*) Position of cranks.

Fig. 22.9

The position of planes of the wheels and cylinders is shown in Fig. 22.9 (*a*), and the position of cranks is shown in Fig 22.9 (*b*). Assuming the plane of wheel *A* as the reference plane, the data may be tabulated as below :

Table 22.3

Plane (1)	*Mass* (*m*) kg (2)	*Radius* (*r*) m (3)	*Cent. force* $\div \omega^2$ (*m.r*) kg-m (4)	*Distance from plane A* (*l*)m (5)	*Couple* $\div \omega^2$ (*m.r.l.*) kg-m^2 (6)
A (*R.P.*)	*B*	*b*	*B.b*	0	0
B	300 *c*	0.3	90 *c*	0.45	40.5 *c*
C	300 *c*	0.3	90 *c*	1.1	99 *c*
D	*B*	*b*	*B.b*	1.55	1.55 *B.b*

Now the couple polygon, to some suitable scale, may be drawn with the data given in Table 22.3 (column 6), as shown in Fig. 22.10. The closing side of the polygon (vector $c'o'$) represents the balancing couple and is proportional to 1.55 *B.b*.

From the couple polygon,

$$1.55\ Bb = \sqrt{(40.5c)^2 + (99c)^2} = 107c$$

$\therefore$ $B.b = 107\ c\ /\ 1.55 = 69\ c$

We know that angular speed,

$$\omega = v/R = 26.8/0.9 = 29.8 \text{ rad/s}$$

$\therefore$ Hammer blow,

$$46 \times 10^3 = B.\ \omega^2 .b$$
$$= 69\ c\ (29.8)^2 = 61\ 275\ c$$

$\therefore$ $c = 46 \times 10^3 / 61\ 275 = 0.751$ **Ans.**

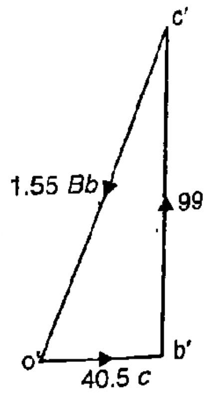


Fig. 22.10

2. Variation in tractive effort

We know that variation in tractive effort

$$= \pm\sqrt{2}(1-c)m.\omega^2.r = \pm\sqrt{2}(1-0.751)\,300(29.8)^2\,0.3$$

$$= 28\ 140 \text{ N} = 28.14 \text{ kN } \textbf{Ans.}$$

3. Maximum swaying couple

We know the maximum swaying couple

$$= \frac{a(1-c)}{\sqrt{2}}\times m.\omega^2.r = \frac{0.65(1-0.751)}{\sqrt{2}}\times 300(29.8)^2\,0.3 = 9148 \text{ N-m}$$

$$= 9.148 \text{ kN-m } \textbf{Ans.}$$

Example 22.5. *The following data apply to an outside cylinder uncoupled locomotive :*

Mass of rotating parts per cylinder = 360 kg ; Mass of reciprocating parts per cylinder = 300 kg ; Angle between cranks = 90° ; Crank radius = 0.3 m ; Cylinder centres = 1.75 m ; Radius of balance masses = 0.75 m ; Wheel centres = 1.45 m.

If whole of the rotating and two-thirds of reciprocating parts are to be balanced in planes of the driving wheels, find :

1. Magnitude and angular positions of balance masses,

2. Speed in kilometres per hour at which the wheel will lift off the rails when the load on each driving wheel is 30 kN and the diameter of tread of driving wheels is 1.8 m, and

3. Swaying couple at speed arrived at in (2) above.

Solution : Given : $m_1 = 360$ kg ; $m_2 = 300$ kg ; $\angle AOD = 90°$; $r_A = r_D = 0.3$ m ; $a = 1.75$ m ; $r_B = r_C = 0.75$ m ; $c = 2/3$.

We know that the equivalent mass of the rotating parts to be balanced per cylinder,

$$m = m_A = m_D = m_1 + c.m_2 = 360 + \frac{2}{3}\times 300 = 560 \text{ kg}$$

1. Magnitude and angular position of balance masses

Let m_B and m_C = Magnitude of the balance masses, and

θ_B and θ_C = angular position of the balance masses m_B and m_C from the crank A.

The magnitude and direction of the balance masses may be determined, graphically, as discussed below :

1. First of all, draw the positions of the planes of the wheels and the cylinders as shown in Fig. 22.11 (*a*). Since the cranks of the two cylinders are at right angles, therefore assuming the position of the cylinder A in the horizontal direction, draw OA and OD at right angles to each other as shown in Fig. 22.11 (*b*).

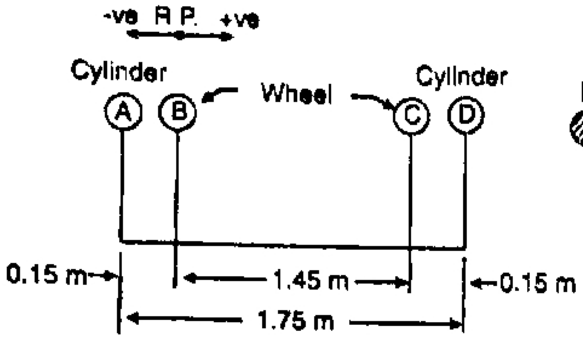
2. Assuming the plane of wheel B as the reference plane, the data may be tabulated as below:

Table 22.4

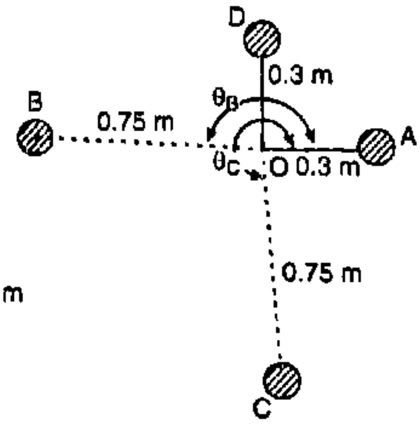
Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent. force ÷ ω^2 (m.r) kg-m (4)	Distance from plane B(l) m (5)	Couple ÷ ω^2 (m.r.l) kg-m² (6)
A	560	0.3	168	– 0.15	– 25.2
B (R.P.)	m_B	0.75	0.75 m_B	0	0
C	m_C	0.75	0.75 m_C	1.45	1.08 m_C
D	560	0.3	168	1.6	268.8

3. Now draw the couple polygon with the data given in Table 22.4 column (6), to some suitable scale as shown in Fig. 22.11(*c*). The closing side $d'o'$ represents the balancing couple and it is proportional to 1.08 m_C. Therefore, by measurement,

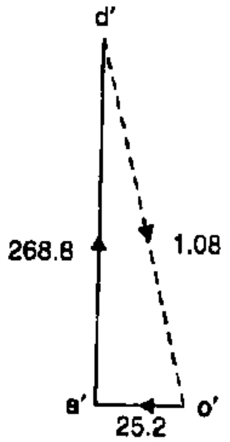
$$1.08\, m_C = 269.6 \text{ kg-m}^2 \quad \text{or} \quad m_C = 249 \text{ kg } \textbf{Ans.}$$



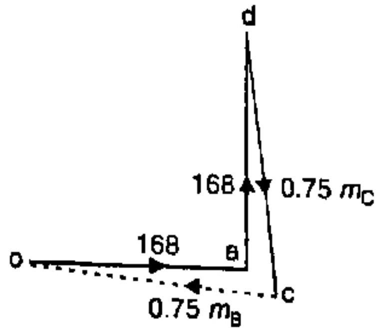
(*a*) Position of planes.



(*b*) Position of masses.



(*c*) Couple polygon.



(*d*) Force polygon.

Fig. 22.11

4. To determine the angular position of the balancing mass C, draw *OC* parallel to vector $d'o'$ as shown in Fig. 22.11 (*b*). By measurement,

$$\theta_C = 275° \textbf{ Ans.}$$

5. In order to find the balancing mass *B*, draw the force polygon with the data given in Table 22.4 column (4), to some suitable scale, as shown in Fig. 22.11 (*d*). The vector *co* represents

the balancing force and it is proportional to 0.75 m_B. Therefore, by measurement,

$$0.75\ m_B = 186.75 \text{ kg-m} \quad \text{or} \quad m_B = 249 \text{ kg } \textbf{Ans.}$$

6. To determine the angular position of the balancing mass *B*, draw *OB* parallel to vector *oc* as shown Fig. 22.11 (*b*). By measurement,

$$\theta_B = 174.5° \textbf{ Ans.}$$

2. Speed at which the wheel will lift off the rails

Given : $P = 30 \text{ kN} = 30 \times 10^3 \text{ N} ; D = 1.8 \text{ m}$

Let ω = Angular speed at which the wheels will lift off the rails in rad/s, and

v = Corresponding linear speed in km/h.

We know that each balancing mass,

$$m_B = m_C = 249 \text{ kg}$$

∴ Balancing mass for reciprocating parts,

$$B = \frac{c\,m_2}{m} \times 249 = \frac{2}{3} \times \frac{300}{560} \times 249 = 89 \text{ kg}$$

We know that $\omega = \sqrt{\frac{P}{B.b}} = \sqrt{\frac{30 \times 10^3}{89 \times 0.75}} = 21.2 \text{ rad/s}$ ...($\because b = r_B = r_C$)

and $v = \omega \times D/2 = 21.2 \times 1.8/2 = 19.08 \text{ m/s}$

$= 19.08 \times 3600/1000 = 68.7 \text{ km/h}$ **Ans.**

3. Swaying couple at speed ω = 21.1 rad/s

We know that the swaying couple

$$= \frac{a(1-c)}{\sqrt{2}} \times m_2.\omega^2.r = \frac{1.75\left[1-\frac{2}{3}\right]}{\sqrt{2}} \times 300(21.2)^2 0.3 \text{ N-m}$$

$$= 16\,687 \text{ N-m} = 16.687 \text{ kN-m } \textbf{Ans.}$$

22.9. Balancing of Coupled Locomotives

The uncoupled locomotives as discussed in the previous article, are obsolete now-a-days. In a coupled locomotive, the driving wheels are connected to the leading and trailing wheels by an outside coupling rod. By such an arrangement, a greater portion of the engine mass is utilised by tractive purposes. In coupled locomotives, the coupling rod cranks are placed diametrically opposite to the adjacent main cranks (*i.e.* driving cranks). The coupling rods together with cranks and pins may be treated as rotating masses

A dynamo converts mechanical energy into electrical energy.

Note : This picture is given as additional information.

Governors

Governor is a device used to maintaining a const mean speed of rotation of the crank-shaft over long periods during which the load on the engine may vary. when the load on the engine increases, the speed of the engine will decrease.

The governor will act in such a way that it will increase the supply of working fluid. similarly when the load on the engine decreases, the speed of the engine increases. Then the governor will act in such a way that the supply of working fluid decreases. Thus the mean speed of rotation of the engine will be maintained constant as closely as possible over a long period.

The function of a flywheel is to limit the fluctuation of speed during each cycle which arises from the fluctuations of turning moment on the crank-shaft. The flywheel doesnot control the speed variations caused by a varying load.

The function of governor is to control the mean speed of rotation over a long period due to the variation of load. The governor has no influence over cyclic speed fluctuations.

Types of Governors

(i) Centrifugal governors and
(ii) Inertia governors.

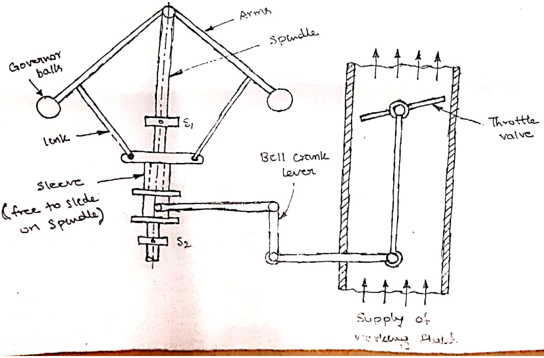
In Centrifugal governors, the two or more masses known as governor balls are caused to revolve about the axis of shaft, which is driven by the engine Crank-shaft through bevel gears. When governor balls are revolving at a uniform speed, the centrifugal force on the balls is equal to the inward controlling force. The inwards controlling force, is provided by a dead weight a spring or a combination of the two.

In case of inertia governors, the governor balls are so arranged that the inertia forces caused by an angular acceleration or retardation of the governor shaft, tend to alter their positions.

Centrifugal Governors

Fig shows the line diagram of a centrifugal governor, which consists of two balls of equal masses (governor balls) attached to the two arms. The upper ends of the arms are pivoted to a spindle, which is driven by the engine through bevel gears.

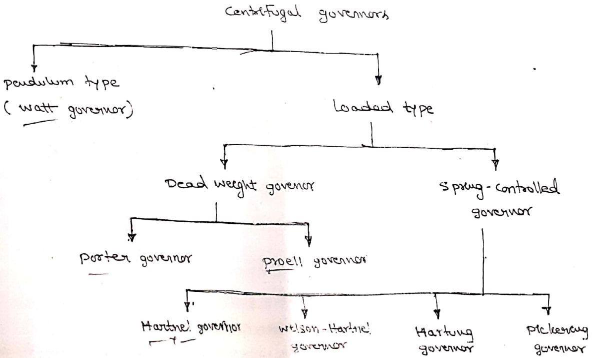
The lower arms (links) are connected to a sleeve which is keyed to the spindle. The sleeve revolves with the spindle, but can slide up and down. The two stops S_1 and S_2 on the spindle prevents the upwards and downward motion of the sleeve. The sleeve is connected by a bell crank lever to a throttle valve, which controls the supply of the working fluid. When sleeve rises, the supply of the working fluid decreases and when sleeve falls, the supply of the working fluid increases.



When the load on the engine decreases, the speed of the engine increases. As the spindle of the governor is driven by the engine, hence the speed of the spindle also increases. This will increase the centrifugal force on the governor balls and the balls will move outwards. Due to the movement of balls outwards the sleeve will rise upwards. The upward movement of the sleeve will operate a throttle valve at the other end of the bell crank lever to reduce the supply of the working fluid by reducing the throttle valve opening.

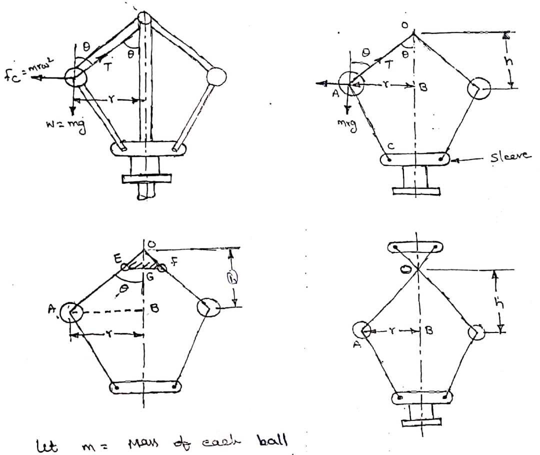
Similarly when the load on the engine increases, the speed of the engine decreases. Also the speed of the spindle of the governor decreases. Hence the centrifugal force on the governor balls will also decrease. The balls of the governor will move inwards and hence the sleeve will move downwards. The downward movement of the sleeve will increase the supply of the working fluid by increasing the opening of the throttle valve and thus the engine speed is increased.

Types of Centrifugal Governors



Watt Governor

Fig shows the simplest form of a Centrifugal governor, which is known as 'Watt governor'. It is the original form of governor used by watt on some of his early steam engines. It consists of a pair of two balls which are attached to the spindle with the help of links or arms. The upper links (arms) are pinned at point O whereas the lower links are fixed to the sleeve which is free to move on the vertical spindle. The spindle is driven by the engine. As the spindle rotates, the balls take up a position depending upon the speed of the spindle.



Let m = Mass of each ball

W = Wt of each ball = m·g

T = Tension in the arm

ω = Angular velocity of the balls, arms and the sleeve

r = Radial distance of the ball centre from spindle-axis i.e radius of the path of rotation of the ball.

fc = Centrifugal force acting on the ball = $\frac{m v^2}{r} = \frac{m\omega^2 r^2}{r} = m\omega^2 r$

h = Height of the governor i.e. the vertical distance from the centre of the ball to the point of intersection of the upper arms along the axis of the spindle.

With the increase of the speed, the height of governor (i.e. h) decreases, whereas with the decrease of the speed, the height h increases.

Assuming the wt. of the arms, links and the sleeve to be negligible as compared to the wt of balls, each ball will be in equilibrium under the action of following forces.

(i) the centrifugal force, f_c acting on the ball where $f_c = m \times \omega^2 \times r$

(ii) the wt of ball, $W = m \times g$

(iii) the tension T in the upper arm.

There will be no tension in the lower link of sleeve is assumed to be massless and also friction is neglected.

Resolving the forces acting on the ball in horizontal direction,

$$T \sin\theta = f_c = m \times \omega^2 \times r \longrightarrow (1)$$

Resolving forces in vertical direction

$$T \cos\theta = W = m \times g \longrightarrow (2)$$

Dividing (1) & (2) we get

$$\frac{T \sin\theta}{T \cos\theta} = \frac{m \times \omega^2 \times r}{m \times g}$$

$$\tan\theta = \frac{\omega^2 \times r}{g} \longrightarrow (3)$$

But from fig we have

$$\tan\theta = \frac{r}{h} \longrightarrow (4)$$

$$\therefore \frac{\omega^2 \times r}{g} = \frac{r}{h}$$

$$h = \frac{r \times g}{\omega^2 \times r} = \frac{g}{\omega^2}$$

If g is taken in m/s^2 and ω in rad/s, then h will be in mts.

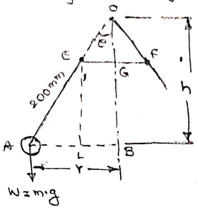
If N is the speed in r.p.m, then $\omega = \frac{2\pi N}{60}$

$$\therefore \quad h = \frac{9.81}{\left(\frac{2\pi N}{60}\right)^2} = 9.81 \times \left(\frac{60}{2\pi}\right)^2 \times \frac{1}{N^2}$$

$$h = \frac{895}{N^2}$$

From Eqn it is clear that height of a watt governor is inversely proportional to the square of the speed. Therefore at high speeds, the value of h is very small. for Ex if N = 50 r.p.m then $h = \frac{895}{50^2}$ = 0.358 m = 35.8 cm. But if N = 300 then $h = \left(\frac{895}{300^2}\right)$ = 0.0099 m = 0.99 cm. Hence this governor works satisfactorily at low speeds ie from 50 to 85 r.p.m.

Determine (i) max speed (ii) min speed (iii) range of speed of a watt governor of open-arm type shown in fig (c) in which length of arm AE = 200 mm and length EF = 30 mm when angle θ changes from 40° to 30°.



length AC = 200 mm, EF = 30 mm,

$\therefore EG = \frac{EF}{2} = \frac{30}{2} = 15$ mm

$\theta = 40°$ & $30°$

$\therefore \theta_1 = 40°$ & $\theta_2 = 30°$

(i) Max speed.

height & speed of governor is $h = \frac{895}{N^2}$

$h = BG + GO$

But $BG = EL = AE\cos\theta = 200\cos\theta$

and OG from triangle OEG, $\tan\theta = \frac{EG}{OG}$

$OG = \frac{EG}{\tan\theta} = \frac{15}{\tan\theta}$

$\therefore h = BG + GO = 200\cos\theta + \frac{15}{\tan\theta}$

$h_1 = \frac{895}{N_1^2}$

$N_1^2 = \frac{895}{h_1}$, when $\theta = 40°$

$h_1 = 200\cos 40° + \frac{15}{\tan 40°} = 171.07$ mm $= 0.17107$ m

$\therefore N_1^2 = \frac{895}{0.17107} = 5231.776$

$N_1 = \sqrt{5231.776} = 72.33$ r.p.m

Min speed (N_2)

$h_2 = 200\cos 30° + \frac{15}{\tan 30°} = 199.18$ mm $= 0.19918$ m

$N_2^2 = \frac{895}{h_2} = \frac{895}{0.19918} = 4493.42$

$N_2 = \sqrt{4493.42} = 67.03$ r.p.m

Range of speed

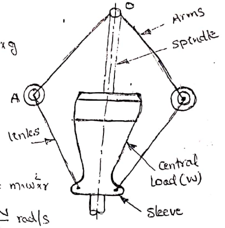
= Max speed − min speed

= $N_1 - N_2$

= $72.33 - 67.03 = 5.3$ r.p.m

Porter Governor

Fig shows the diagram of a porter governor. In case of Porter governor, a central heavy load is attached to the sleeve. The central load and sleeve moves up and down the spindle.



Let M = Mass of central load

W = weight of central load = $M \times g$

w = wt of each ball = $m \times g$

m = Mass of each ball

h = Height of governor

r = Radius of rotation

F_c = Centrifugal force on the ball = $m\omega^2 r$

ω = Angular speed of ball = $\frac{2\pi N}{60}$ rad/s

N = Speed of ball in r.p.m

T_1 = Tension in upper arm

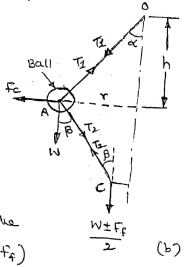
T_2 = Tension in lower link

α = Angle of inclination of the upper arm to the vertical

β = Angle of inclination of the lower link to the vertical

F_f = force of friction b/w sleeve and spindle.

The force of friction always acts in a direction opp to that of the motion. When sleeve moves up the force of friction acts in the downward direction. Then the total force acting on the sleeve in the downward direction will be $(W + f_f)$. Similarly when the sleeve moves down, the total force on the sleeve will be $(W - f_f)$. In general, the total force acting on the sleeve will be $(W \pm f_f)$ depending upon whether the sleeve moves upwards or downwards.



The relation b/w the height of governor and angular speed of ball

Fig(b) shows the forces acting on left-hand half of the governor i.e. on the sleeve and on each ball.

Firstly, considering the equilibrium of left-half of sleeve,

$$T_2 \cos\beta = \frac{W \pm f_f}{2}$$

or

$$T_2 = \frac{W \pm f_f}{2 \cos\beta} \longrightarrow (1)$$

Now, considering the equilibrium of left ball,

Resolve the forces vertically,

$$T_1 \cos\alpha = w + T_2 \cos\beta \longrightarrow (2)$$

Resolve the forces horizontally,

$$T_1 \sin\alpha + T_2 \sin\beta = f_c \longrightarrow (3)$$

or

$$T_1 \sin\alpha + \left(\frac{W \pm f_f}{2 \cos\beta}\right) \sin\beta = f_c$$

$$T_1 \sin\alpha = f_c - \left(\frac{W \pm f_f}{2}\right) \tan\beta \longrightarrow (4)$$

The tension T_1 can be eliminated from eqns (2) & (4)

Dividing (4) by (2)

$$\frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{f_c - \left(\frac{W \pm f_f}{2}\right)\tan\beta}{w + T_2 \cos\beta}$$

$$\tan\alpha = \frac{f_c - \left(\frac{W \pm f_f}{2}\right)\tan\beta}{w + \left(\frac{W \pm f_f}{2 \cos\beta}\right)\cos\beta}$$

or

$$\left(w + \frac{W \pm f_f}{2}\right)\tan\alpha = f_c - \left(\frac{W \pm f_f}{2}\right)\tan\beta$$

or

$$\left(w + \frac{W \pm f_f}{2}\right) = \frac{f_c}{\tan\alpha} - \left(\frac{W \pm f_f}{2}\right)\frac{\tan\beta}{\tan\alpha}$$

let $\frac{\tan\beta}{\tan\alpha} = k$

$$\left(w + \frac{W \pm f_f}{2}\right) = \frac{f_c}{\tan\alpha} - \left(\frac{W \pm f_f}{2}\right) \times k$$

But from fig (b), $\tan\alpha = \frac{r}{h}$

$$\left(w + \frac{W \pm f_f}{2}\right) = \frac{F_c \times h}{r} - \left(\frac{W \pm f_f}{2}\right) \times k$$

$$= \frac{m \times \omega^2 \times r \times h}{r} - \left(\frac{W \pm f_f}{2}\right) \times k$$

$$\left(m \times g + \frac{M \times g \pm f_f}{2}\right) = m \times \omega^2 \times h - \left(\frac{M \times g \pm f_f}{2}\right) \times k$$

$$m \times g + \frac{M \times g \pm f_f}{2} + \left(\frac{M \times g \pm f_f}{2}\right) \times k = m \times \omega^2 \times h$$

$$m \times g + \frac{M \times g \pm f_f}{2}(1+k) = m \times \omega^2 \times h$$

$$\boxed{\omega^2 = \frac{m \times g + \left(\frac{M \times g \pm f_f}{2}\right)(1+k)}{mh}}$$

$$= \frac{m \times g + \left(\frac{M \times g \pm f_f}{2}\right)(1+k)}{m \times g} \times \frac{g}{h}$$ (multiplying & dividing by g)

$$\omega^2 = \frac{w + \left(\frac{W \pm f_f}{2}\right)(1+k)}{w} \times \frac{g}{h}$$

the height of governor h is obtained as

$$h = \frac{m \times g + \left(\frac{M \times g \pm f_f}{2}\right)(1+k)}{m \times \omega^2}$$

If there is no friction b/w sleeve and spindle, then $f_f = 0$, Then

$$\omega^2 = \frac{m \times g + \frac{M \times g}{2}(1+k)}{m \times h}$$

$$\omega^2 = \frac{w + \frac{W}{2}(1+k)}{w} \times \frac{g}{h}$$

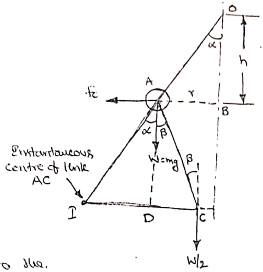
Instantaneous Centre Method

In this method, the equilibrium of the lower arm AC is considered. The forces acting on the lower arm AC are

(i) Centrifugal force F_c through A

(ii) The wt of ball (m×g) through A, and

(iii) Half of the w.t of the sleeve ie $\frac{W}{2}$



First, the instantaneous centre of the lower arm AC is obtained. As the point A moves along a circular arc which has O as centre and AO is radius and point C moves parallel to the axis of the governor, the instantaneous centre I lies at the point of intersection of OA produced and a line drawn through C perpendicular to the governor axis.

Taking moments of forces (ie F_c, mg and $\frac{W}{2}$) acting on lower arm AC, about the point I.

$$F_c \times AD = (m\times g)\times ID + \frac{W}{2}\times IC$$

$$F_c = (m\times g)\times \frac{ID}{AD} + \frac{W}{2}\times\frac{IC}{AD}$$

$$= m\times g\times\tan\alpha + \frac{W}{2}\times\left(\frac{ID+CD}{AD}\right)$$

$$= m\times g\times\tan\alpha + \frac{W}{2}\times\left(\frac{ID}{AD}+\frac{CD}{AD}\right)$$

$$= m\times g\times\tan\alpha + \frac{M\times g}{2}(\tan\alpha+\tan\beta)$$

$$= m\times g\times\tan\alpha + \frac{M\times g}{2}\tan\alpha\left(1+\frac{\tan\beta}{\tan\alpha}\right)$$

$$= m\times g\times\tan\alpha + \frac{M\times g}{2}\tan\alpha(1+k)$$

$$= \tan\alpha\left[m\times g + \frac{M\times g}{2}(1+k)\right]$$

But from triangle OAB, $\tan\alpha = \frac{r}{h}$ and $F_c = m \times \omega^2 \times r$

$$m \times \omega^2 \times r = \frac{r}{h}\left[m \times g + \frac{M \times g}{2}(1+k)\right]$$

$$\omega^2 = \frac{m \times g + \frac{M \times g}{2}(1+k)}{m \times h}$$

If $k = 1$ which is true when $\tan\alpha = \tan\beta$

$$\omega^2 = \frac{m \times g + M \times g}{m \times h}$$

$$= \frac{(m+M)g}{m \times h}$$

But $\omega = \frac{2\pi N}{60}$

$$\left(\frac{2\pi N}{60}\right)^2 = \frac{(m+M)g}{m \times h}$$

$$h = \frac{(m+M)g}{m \times \left(\frac{2\pi N}{60}\right)^2}$$

$$= \frac{(m+M) \times g \times 3600}{m \times 4\pi^2 \times N^2}$$

$$h = \frac{m+M}{m} \times \frac{9.81 \times 3600}{4\pi^2 \times N^2}$$

$$h = \frac{m+M}{m} \times \frac{894.56}{N^2}$$

h is in meter.

If frictional force at the sleeve is taken into consideration, then total force in general acting on C when sleeve moves upwards or downwards is equal to $\frac{1}{2}(M \times g \pm F)$. Then

$$\omega^2 = \frac{m \times g + \frac{(M \times g \pm F)}{2}(1+k)}{m \times h}$$

$$\therefore \quad F_{C_2} = \frac{107791.7}{320.56} = 336.26$$

But $\quad F_{C_2} = m \times \omega_2^2 \times r_2 = 3.75 \times \omega_2^2 \times 0.2642 \quad (\because \; r_2 = 264.2 \text{ mm} = 0.2642 \text{ m})$

$$\therefore \quad 336.26 = 3.75 \times \omega_2^2 \times 0.2642$$

or $$\omega_2 = \sqrt{\frac{336.26}{3.75 \times 0.2642}} = 18.42 \text{ rad/s}$$

and $$N_2 = \frac{60 \times \omega_2}{2\pi} = \frac{60 \times 18.42}{2\pi} = \mathbf{175.92 \text{ r.p.m.}} \quad \textbf{Ans.}$$

15.7. Hartnell Governor

Fig. 15.12 shows a Hartnell governor, which is a spring loaded governor. Two bell-crank levers, each carrying a ball at one end and a roller at the other, are pivoted at points O and O' to the frame. The rollers fit into a groove in the sleeve. The frame is attached to the governor spindle and hence rotates with it. A helical spring in compression provides equal downward forces on the two rollers through a collar on the sleeve.

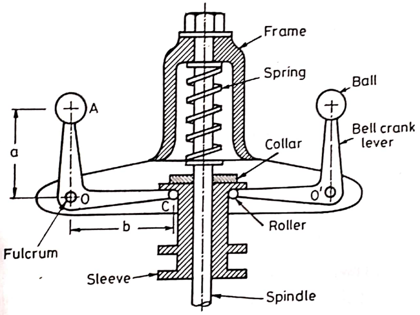


Fig. 15.12

When the speed increases the radius of rotation of balls increases and the balls move away from the spindle axis. The bells are connected to the bell-crank levers which are pivoted at points O and O'. As the balls move away from the spindle axis, the rollers (connected at the other end of the bell-crank lever) lift the sleeve against the spring force. If the speed decreases, the sleeve moves downwards. The movement of the sleeve is transferred to the throttle of the engine to control the amount of energy supplied to the engine.

Let r_1 = Minimum radius of rotation of ball centre from spindle axis,

r_2 = Maximum radius of rotation of ball centre from spindle axis,

S_1 = Spring force exerted on sleeve at minimum radius,

S_2 = Spring force exerted on sleeve at maximum radius,

m = Mass of each ball,

M = Mass of sleeve,

N_1 = Minimum speed of governor at minimum radius,

N_2 = Maximum speed of governor at maximum radius,

ω_1 and ω_2 = Corresponding minimum and maximum angular velocities

$(F_c)_1$ = Centrifugal force corresponding to minimum speed = $m \times \omega_1^2 \times r_1$

$(F_c)_2$ = Centrifugal force corresponding to maximum speed = $m \times \omega_2^2 \times r_2$

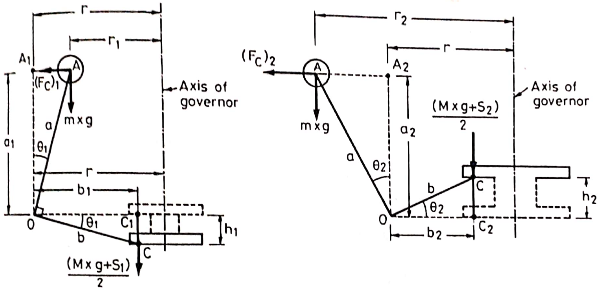
s = Stiffness of spring or the force required to compress the spring by one mm,

r = Distance of fulcrum O from the governor axis or radius of rotation when the governor is in mid-position,

a = Length of ball arm of bell-crank lever *i.e.* distance OA

b = Length of sleeve arm of bell-crank lever *i.e.* distance OC.

Figs. 15.13 (*a*) and 15.13 (*b*) shows the forces acting on the bell-crank lever in two positions *i.e.* at minimum radius position and at maximum radius position.



(*a*) Position of minimum radius (*b*) Position of maximum radius

Fig. 15.13

Let h = compression of the spring when radius of rotation changes from r_1 to r_2. This is also known as lift of the sleeve.

(*i*) *Position of minimum radius* (Refer to Fig. 15.13 (*a*)).

The position of bell-crank lever at the minimum radius is shown by AOC whereas the position of bell-crank lever when governor is in mid-position is shown by dotted line A_1OC_1.

Let h_1 = lift of sleeve *i.e.* vertical distance CC_1.

The angle turned by bell-crank lever between mid-position and minimum radius position is θ_1. This means the angle between OA and OA_1 is same as between OC and OC_1

$$\therefore \quad \theta_1 = \frac{CC_1}{OC} = \frac{AA_1}{OA} \qquad \left(\because \ \theta_1 = \frac{\text{Arc}}{\text{Radius}}. \text{ For } OCC_1, \text{ radius is } OC \text{ and arc is } CC_1\right).$$

or
$$\frac{CC_1}{OC} = \frac{AA_1}{OA}$$

or
$$\frac{h_1}{b} = \frac{(r - r_1)}{a} \qquad (\because \ AA_1 = r - r_1 \,;\, OA = a \text{ and } OC = b)$$

or
$$h_1 = \frac{b}{a}(r - r_1) \qquad \ldots(i)$$

(*ii*) *Position of maximum radius* (Refer to Fig. 15.13 (*b*))

The position of the bell-crank lever at the maximum radius is shown by AOC whereas the position of bell-crank when governor is in mid-position is shown by dotted line A_2OC_2.

Let h_2 = lift of sleeve from mid-position *i.e.* vertical distance C_2C.

The angle turned by bell-crank lever between mid-position and maximum radius position is θ_2, *i.e.* $\angle C_2OC = \angle A_2OA = \theta_2$.

$$\therefore \qquad \theta_2 = \frac{h_2}{OC} = \frac{AA_2}{OA}$$

or $$\frac{h_2}{b} = \frac{(r_2 - r)}{a} \qquad (\because AA_2 = r_2 - r_1)$$

$$\therefore \qquad h_2 = \frac{b}{a}(r_2 - r) \qquad ...(ii)$$

Adding equations (*i*) and (*ii*), we get

$$h_1 + h_2 = \frac{b}{a}(r - r_1) + \frac{b}{a}(r_2 - r)$$

$$= \frac{b}{a}r - \frac{b}{a}r_1 + \frac{b}{a}r_2 - \frac{b}{a}r = \frac{b}{a}(r_2 - r_1)$$

or $$h = \frac{b}{a}(r_2 - r_1) \qquad (\because h = h_1 + h_2 = \text{total lift}) \qquad ...(iii) \quad ...(15.10)$$

(*iii*) *Position of minimum radius* (Refer to Fig. 15.13 (*a*))

Taking moments of all forces about fulcrum O, we get

$$\frac{(M \times g + S_1)}{2} \times b_1 + m \times g \times AA_1 = (F_C)_1 \times a_1 \qquad ...(A)$$

or $$\frac{(M \times g + S_1)}{2} \times b_1 = (F_C)_1 \times a_1 - m \times g \times AA_1$$

or $$(M \times g + S_1) = \frac{2}{b_1}[(F_C)_1 \times a_1 - m \times g \times AA_1]$$

$$= \frac{2}{b_1}[(F_C)_1 \times a_1 - mg(r - r_1)] \qquad (\because AA_1 = r - r_1) \qquad ...(iv)$$

(*iv*) *Position of maximum radius* (Refer to Fig. 15.13 (*b*))

Taking moments of all forces about the fulcrum, we get

$$\frac{(M \times g + S_2)}{2} \times b_2 = (F_C)_2 \times a_2 + mg \times AA_2 \qquad ...(B)$$

$$= (F_C)_2 \times a_2 + mg(r_2 - r) \qquad (\because AA_2 = r_2 - r)$$

or $$(M \times g + S_2) = \frac{2}{b_2}[(F_C)_2 \times a_2 + mg\,(r_2 - r)] \qquad ...(v)$$

Substracting equation (*iv*) from equation (*v*), we get

$$S_2 - S_1 = \frac{2}{b_2}[(F_C)_2 \times a_2 + mg\,(r_2 - r)] - \frac{2}{b_1}[(F_C)_1 \times a_1 - mg(r - r_1)]$$

But spring stiffness(s) is given by

$$s = \frac{S_2 - S_1}{\text{Total lift}} = \frac{S_2 - S_1}{h}$$

But from equation (15.10), $h = \frac{b}{a}(r_2 - r_1)$

$$\therefore \quad s = \frac{S_2 - S_1}{\frac{b}{a}(r_2 - r_1)} = \frac{a}{b}\left[\frac{S_2 - S_1}{r_2 - r_1}\right] \quad \text{...(15.11)}$$

Value of spring stiffness if obliquity of the arms of bell-crank lever is neglected and also the moment due to the weight of the balls is neglected.

(*i*) If obliquity of the arms is neglected, then $b_1 = b_2 = b$ and $a_1 = a_2 = a$.

(*ii*) If the moment due to weight of the ball is neglected, then $mg \times AA_1 = 0$ and $m \times g \times AA_2 = 0$.

Substituting these values (*i.e.* $b_1 = b_2 = b$; $a_1 = a_2 = a$ and $m \times g \times AA_1 = m \times g \times AA_2 = 0$) in above equations (A) and (B), we get

$$\left(\frac{M \times g + S_1}{2}\right) \times b + 0 = (F_C)_1 \times a$$

and

$$\frac{M \times g + S_2}{2} \times b + 0 = (F_C)_2 \times a$$

or

$$M \times g + S_1 = \frac{2a}{b} \times (F_C)_1 \quad \text{...(vi)}$$

and

$$M \times g + S_2 = \frac{2a}{b}(F_C)_2 \quad \text{...(vii)}$$

Substracting equation (*vi*) from equation (*vii*), we get

$$S_2 - S_1 = \frac{2a}{b}[(F_C)_2 - (F_C)_1] \quad \text{...(viii)}$$

But spring stiffness s is given by,

$$s = \frac{S_2 - S_1}{h} \text{ where } h = \frac{b}{a}(r_2 - r_1)$$

$$= \frac{\left(\frac{2a}{b}\right)[(F_C)_2 - (F_C)_1]}{\frac{b}{a}(r_2 - r_1)} \qquad \left(\because \; S_2 - S_1 = \frac{2a}{b}[(F_C)_2 - (F_C)_1]\right)$$

$$= 2\left(\frac{a}{b}\right)^2\left[\frac{(F_C)_2 - (F_C)_1}{(r_2 - r_1)}\right] \quad \text{...(C)}$$

The stiffness of the given spring is constant for all positions. Hence stiffness of the spring for minimum and intermediate positions can be obtained from equation (C) by substituting $(F_C)_2 = F_C$ and $r_2 = r$ as for intermediate position, the centrifugal force is F_C and radius is r.

$$\therefore \quad s = 2\left(\frac{a}{b}\right)^2\left[\frac{F_C - (F_C)_1}{r - r_1}\right] \quad \text{...(D)}$$

Similarly the spring stiffness for intermediate and maximum position is obtained from equation (C) by substituting $(F_C)_1 = F_C$ and $r_1 = r$.

$$\therefore \quad s = 2\left(\frac{a}{b}\right)^2\left[\frac{(F_C)_2 - F_C}{r_2 - r}\right] \quad \text{...(E)}$$

The three values of s given by equation (C), (D) and (E) can be equated as

$$2\left(\frac{a}{b}\right)^2\left[\frac{(F_C)_2-(F_C)_1}{r_2-r_1}\right]=2\left(\frac{a}{b}\right)^2\left[\frac{F_C-(F_C)_1}{r-r_1}\right]=2\left(\frac{a}{b}\right)^2\left[\frac{(F_C)_2-F_C}{r_2-r}\right]$$

or

$$\left[\frac{(F_C)_2-(F_C)_1}{r_2-r_1}\right]=\frac{F_C-(F_C)_1}{r-r_1}=\frac{(F_C)_2-F_C}{r_2-r} \quad \text{...(F)}$$

From first two equations, we have

$$F_C-(F_C)_1=[(F_C)_2-(F_C)_1]\left[\frac{r-r_1}{r_2-r_1}\right]$$

or

$$F_C=(F_C)_1+[(F_C)_2-(F_C)_1]\left[\frac{r-r_1}{r_2-r_1}\right] \quad \text{...(15.12)}$$

Similarly from 1st and last part of equation (F), we have

$$F_C=(F_C)_2-[(F_C)_2-(F_C)\left[\frac{r_2-r}{r_2-r_1}\right].$$

Note. 1. The weight of sleeve $M \times g$ is replaced by $(M \times g \pm F)$ when friction is taken into account.

2. The obliquity effect of the arms and moment due to the weight of the balls is neglected, unless otherwise stated.

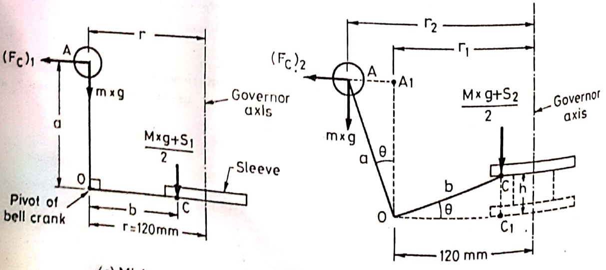
Problem 15.10. *A Hartnell governor having a central sleeve spring and two right-angled bell crank levers operates between 290 r.p.m. and 310 r.p.m. for a sleeve lift of 15 mm. The sleeve arms and the ball arms are 80 mm and 120 mm respectively. The levers are pivoted at 120 mm from the governor axis and mass of each ball is 2.5 kg. The ball arms are parallel to the governor axis at the lowest equilibrium speed. Determine: (i) loads on the spring at the lowest and the highest equilibrium speeds and (ii) stiffness of the spring.*

(AMIE, S 1978)

Sol. Given :

$N_1 = 290$ r.p.m. ; $N_2 = 310$ r.p.m. $\therefore$ $\omega_1 = \frac{2\pi N_1}{60} = \frac{2\pi \times 290}{60} = 30.4$ rad/s and $\omega_2 = \frac{2\pi \times 310}{60} = 32.5$ rad/s, $h = 15$ mm ; sleeve arm of bell crank lever, $b = 80$ mm ; ball arm, $a = 120$ mm ; distance of pivot of lever from governor axis, $r = 120$ mm, $m = 2.5$ kg.

Find : (*i*) Load on springs *i.e.* S_1 and S_2 and (*ii*) Stiffness of spring(s).



(*a*) Minimum radius (*b*) Maximum radius

Fig. 15.14

The ball arms are parallel to the governor axis at the lowest equilibrium speed of 290 r.p.m. as shown in Fig. 15.14 (*a*), hence $r_1 = r = 120$ mm $= 0.12$ m.

(*i*) *Loads on the spring at the lowest and highest equilibrium speeds*

Let S_1 = Spring load at the lowest equilibrium speed, and

S_2 = Spring load at the highest equilibrium speed.

The centrifugal force at the lowest equilibrium speed,

$$(F_C)_1 = m \times \omega_1^2 \times r_1 = 2.5 \times (30.4)^2 \times 0.12 = 277 \text{ N}$$

The centrifugal force at the highest equilibrium speed,

$$(F_C)_2 = m \times \omega_2^2 \times r_2 = 2.5 \times 32.5^2 \times r_2 \qquad ...(i)$$

In the above equation, the value of r_2 is unknown. This value is obtained by considering the position of ball arm and sleeve arm at the highest equilibrium speed as shown in Fig. 15.14 (*b*).

In Fig. 15.14 (*b*), the triangles OCC_1 and OAA_1 are similar (the angle turned by bell-crank lever *i.e.* θ is same)

$$\therefore \quad \frac{CC_1}{OC} = \frac{AA_1}{OA} \quad \text{or} \quad \frac{h}{b} = \frac{(r_2 - r_1)}{a}$$

or $$h = (r_2 - r_1) \times \frac{b}{a}$$

$$r_2 = \frac{a}{b} \times h + r_1 = \frac{120}{80} \times 15 + 120 = 142.5 \text{ mm} \quad \text{or} \quad 0.1425 \text{ m.}$$

Substituting the value of r_2 in equation (*i*), we get

$$(F_C)_2 = 2.5 \times 32.5^2 \times 0.1425 = 376 \text{ N.}$$

Case I. Taking moments about O for the lowest equilibrium speed, (Refer to Fig. 15.14 (*a*)), we have

$$(F_C)_1 \times a + mg \times 0 = \left(\frac{M \times g + S_1}{2}\right) \times b$$

or $$(F_C)_1 \times a = \frac{S_1}{2} \times b \qquad (\because \; M = 0)$$

or $$S_1 = 2 \times (F_C)_1 \times \frac{a}{b}$$

$$= 2 \times 277 \times \frac{120}{80} = \mathbf{831 \text{ N.}} \quad \textbf{Ans.} \qquad (\because \; (F_C)_1 = 277)$$

Case II. Taking moments about O for the highest equilibrium speed, (Refer to Fig. 15.14 (*b*)), we have

$$(F_C)_2 \times a + mg \times AA_1 = \frac{(M \times g + S_2)}{2} \times b$$

$$376 \times 0.12 + (2.5 \times 9.81) \times 0.0225 = \frac{S_2}{2} \times 0.08$$

$(\because \; AA_1 = r_2 - r_1 = 142.5 - 120 = 22.5 \text{ mm} = 0.0225 \text{ m};$
$a = 120 \text{ mm} = 0.12 \text{ m and } b = 80 \text{ mm} = 0.08 \text{ m})$

or $$45.12 + 0.552^* = S_2 \times 0.04$$

or $$45.672 = 0.04 \times S_2$$

or $$S_2 = \frac{45.672}{0.04} = \mathbf{1141.8 \text{ N.}} \quad \textbf{Ans.}$$

*The moment due to the weight of ball is $m \times g \times AA_1 = 0.552$ N. This is very small in comparison to the moment due to $(F_C)_1$ which is 45.12 N.

(ii) Stiffness of the spring(s)

The stiffness of the spring is given by,

$$s = \frac{S_2 - S_1}{\text{Sleeve lift}} = \frac{S_2 - S_1}{h} = \frac{1141.8 - 831}{15} = \mathbf{20.72\ N/mm.}\quad \textbf{Ans.}$$

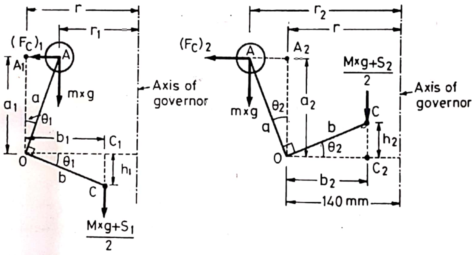
Problem 15.11. *In a spring loaded Hartnell type of governor, the mass of each ball is 4 kg and the lift of the sleeve is 50 mm. The governor begins to float at 240 r.p.m., when radius of the ball path is 110 mm. The mean working speed of the governor is 20 times the range of the speed when friction is neglected. The lengths of the ball and roller arms of the bell-crank lever are 120 mm and 100 mm respectively. The pivot centre and the axis of the governor are 140 mm apart. Determine the initial compression of the spring, taking into account the obliquity of arms.*

If the friction is equivalent to a force of 20 N at the sleeve, find the total alteration in speed before the sleeve begins to move from mid-position.

Sol. Given :

$m = 4$ kg ; $h = 50$ mm $= 0.05$ m ; $N_1 = 240$ r.p.m. or $\omega_1 = \frac{2\pi N_1}{60} = \frac{2\pi \times 240}{60} = 8\pi$ rad/s ;

$r_1 = 110$ mm $= 0.11$ m ; mean speed $= 20 \times$ range of speed ; $a = 120$ mm $= 0.12$ m ; $b = 100$ mm $= 0.1$ m ; $r = 140$ mm $= 0.14$ m ; $F = 20$ N.



(*a*) Position of minimum radius. (*b*) Position of maximum radius.

Fig. 15.15

Let N_1 = Minimum speed at minimum radius, r_1

N_2 = Maximum speed at maximum radius, r_2.

We know that mean speed, $N = \frac{N_1 + N_2}{2}$ and range of speed $= N_2 - N_1$

But mean speed $= 20 \times$ range of speed (given)

$\therefore$ $N = 20 \times (N_2 - N_1)$

or $\frac{N_1 + N_2}{2} = 20 \times (N_2 - N_1)$

or $N_1 + N_2 = 40(N_2 - N_1) = 40N_2 - 40N_1$

or $41N_1 = 39N_2$

or $N_2 = \frac{41 \times N_1}{39} = \frac{41 \times 240}{39} = 252.3$ r.p.m.

Now taking the moments about fulcrum for maximum radius position,

$$(F_C)_2 \times a = \left(\frac{M \times g + S_2 + F}{2}\right) \times b$$

[For maximum radius, sleeve moves upwards and frictional force acts downwards]

or $$(F_C)_2 = \frac{M \times g + S_2 + F}{2} \qquad (\because a = b)$$

or $$1179.87 = \frac{5 \times 9.81 + S_2 + 35}{2} \qquad [\because (F_C)_2 = 1179.87]$$

or $$S_2 = 2 \times 1179.87 - 5 \times 9.81 - 35 = 2359.74 - 49.05 - 35 = 2275.69 \text{ N}$$

$\therefore$ Stiffness, $s = \dfrac{S_2 - S_1}{h} = \dfrac{2275.69 - 1222.65}{30} = $ **35.1 N/mm. Ans.**

(iii) Initial compression of the spring

Initial compression of the spring is $= \dfrac{S_1}{s} = \dfrac{1222.65}{35.1} =$ **34.83 mm. Ans.**

15.8. Wilson-Hartnell Governor

Fig. 15.17 shows a Wilson-Hartnell governor which is a spring loaded type of governor. In this governor, the balls are connected by two springs which are known as main springs. The main springs are arranged symmetrically on either side of the sleeve. The balls are attached to the vertical arms of the two bell-crank levers. The horizontal arms of the bell-crank levers carry two rollers at their ends. The rollers at the horizontal arm press against the sleeve. *The bell-cranks rotate with the spindle.* When speed increases, the ball-radius increases, the springs exert an inward pull *P* on the balls and the rollers press against the sleeve which is raised.

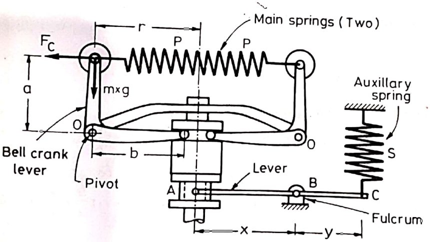


Fig. 15.17

An adjustable auxiliary spring (*S*) is attached to the sleeve through a lever. The lever is pivoted at a fulcrum *B*. One end of the lever is connected to the auxiliary spring whereas the other end of the lever fits into the groove in the sleeve. The auxiliary spring tends to keep the sleeve down.

Let m = Mass of each ball

M = Mass of sleeve

W = Weight of sleeve = $M \times g$

P = Tension (or Pull) in the main spring

S = Tension in the auxiliary spring

F_C = Centrifugal force of each ball

r = Radius of rotation of balls

s = Stiffness of each ball spring

s^* = Stiffness of auxiliary spring.

The total downward force on the sleeve

= Weight of sleeve + force at lever end A due to tension S in auxiliary spring (*i.e.* at point C)

$$= W + \frac{S \times y}{x} = \left(M \times g + \frac{S \times y}{x}\right)$$

Taking the moments about the pivot O of the bell-crank lever and neglecting the effect of the pull of gravity on the balls, we have

$$(F_C - P) \times a = \left(\frac{W + \frac{S \times y}{x}}{2}\right) \times b \quad \text{...(i)}$$

Let corresponding to minimum speed, F_{C_1} = Centrifugal force $= m \times \omega_1^2 \times r_1$

P_1 = Tension in main spring, S_1 = Tension in auxiliary spring

and F_{C_2}, P_2 and S_2 = corresponding values of centrifugal force, tension in main spring and tension in auxiliary spring corresponding to maximum speed.

Substituting these values in equation (*i*), we have for minimum speed

$$(F_{C_1} - P_1) \times a = \frac{\left(M \times g + \frac{S_1 \times y}{x}\right) \times b}{2} \quad (\because \; W = m \times g) \quad \text{...(ii)}$$

Similarly for maximum speed, we have

$$(F_{C_2} - P_2) \times a = \frac{\left(M \times g + \frac{S_2 \times y}{x}\right) \times b}{2} \quad \text{...(iii)}$$

Subtracting equation (*ii*) from equation (*iii*), we have

$$[(F_{C_2} - F_{C_1}) - (P_1 - P_2)] \times a = (S_2 - S_1) \times \frac{y}{x} \times \frac{b}{2} \quad \text{...(iv)}$$

When the radius increases from r_1 to r_2, the ball springs will be extended by the amount $(d_2 - d_1)$ or $2(r_2 - r_1)$ and auxiliary spring will be extended* by the amount $(r_2 - r_1)\frac{b}{a} \times \frac{y}{x}$. The main spring consists of two springs.

$\therefore$ $P_2 - P_1$ = Net pull (or tension) in two main spring when radius increases from r_1 to r_2

= 2 × Force exterted by each main spring

= 2 × [stiffness of main spring × extension of ball springs]

$= 2 \times [s \times 2(r_2 - r_1)]$

$= 4 \times s \times (r_2 - r_2) = 4 \, . \, s \, . \, (r_2 - r_1)$

*Let h = total lift of sleeve when radius increases from r_1 to r_2. The angle turned by ball arm and sleeve arm are same. Hence $\frac{h}{b} = \frac{(r_2 - r_1)}{a}$ or $h = \frac{b}{a}(r_2 - r_1)$. But h is the lift of lever at point A. The point C will move down. Let h^* = downward movement of C [See Fig. 15.17 (*b*)]. Then $\frac{h}{x} = \frac{h^*}{y}$ or $h^* = h \times \frac{y}{x} = \frac{b}{a}(r_2 - r_1) \times \frac{y}{x}$.

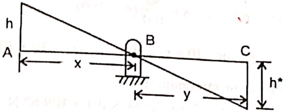


Fig. 15.17 (b)

The net force in auxiliary spring is given by

$$S_2 - S_1 = \text{stiffness of auxiliary spring} \times \text{extension of auxiliary spring}$$

$$= s^* \cdot (r_2 - r_1) \cdot \frac{b}{a} \cdot \frac{y}{x}$$

Substituting the values of $(P_2 - P_1)$ and $(S_2 - S_1)$ in equation (iv), we get

$$[(F_{C_2} - F_{C_1}) - 4 \cdot s \cdot (r_2 - r_1)] \cdot a = \left[s^* \cdot (r_2 - r_1) \cdot \frac{b}{a} \cdot \frac{y}{x} \right] \cdot \frac{y}{x} \cdot \frac{b}{2}$$

or

$$(F_{C_2} - F_{C_1}) - 4 \cdot s \cdot (r_2 - r_1) = s^* \cdot (r_2 - r_1) \cdot \frac{b}{a} \cdot \frac{y}{x} \cdot \frac{y}{x} \cdot \frac{b}{2} \cdot \frac{1}{a}$$

$$= \frac{s^*}{2} \cdot (r_2 - r_1) \cdot \left(\frac{b}{a} \cdot \frac{y}{x} \right)^2$$

Dividing by $(r_2 - r_1)$ to both sides, we get

$$\frac{F_{C_2} - F_{C_1}}{r_2 - r_1} - 4 \cdot s = \frac{s^*}{2} \cdot \left(\frac{b}{a} \cdot \frac{y}{x} \right)^2$$

or

$$\frac{F_{C_2} - F_{C_1}}{r_2 - r_1} = 4 \cdot s + \frac{s^*}{2} \left(\frac{b}{a} \cdot \frac{y}{x} \right)^2 \quad ...(15.13)$$

If auxiliary spring is not used, then $s^* = 0$, then

$$\frac{F_{C_2} - F_{C_1}}{r_2 - r_1} = 4 \cdot s \quad ...(15.14)$$

Problem 15.14. *The mass of each ball in a Wilson-Hartnell type of governor is 2.5 kg. The length of ball arm of each bell-crank lever is 100 mm whereas the length of the sleeve arm of bell-crank lever is 80 mm. The minimum equilibrium speed is 200 r.p.m. when radius of rotation is 100 mm. When the sleeve is lifted by 8 mm, the equilibrium speed is 212 r.p.m. The stiffness of each of the springs connected to the balls is 200 N/m. The lever for the auxiliary spring is pivoted at the mid-point. Find the stiffness of the auxiliary spring.*

Sol. Given :

$m = 2.5$ kg ; $a = 100$ mm $= 0.1$ m ; $b = 80$ mm $= 0.08$ m ; $N_1 = 200$ r.p.m.

or $\omega_1 = \frac{2\pi N_1}{60} = \frac{2\pi \times 200}{60} = 20.94$ rad/s ; $r_1 = 100$ mm $= 0.1$ m ; $h = 8$ mm ; $N_2 = 212$ r.p.m

or $\omega_2 = \frac{2\pi N_2}{60} = \frac{2\pi \times 212}{60} = 22.2$ rad/s ; $s = 200$ N/m ; lever of auxiliary spring is pivoted at mid-point hence $x = y$; $s^* = ?$

Let us first find the values of centrifugal forces. The radius of rotation r_1 is known, but r_2 is unknown. The radius r_2 will be obtained by using,

[See equation (15.10)]

$$\text{lift} = \frac{b}{a}(r_2 - r_1)$$

15.9. Some Important Definitions

Let us define the followings most important terms which are used in connection with governors :

(i) Sensitiveness (ii) Stability

(iii) Isochronism and (iv) Hunting.

15.9.1. Sensitiveness

A governor is said to be sensitive if with a given fractional change of speed, the displacement of the sleeve is bigger. Hence the movement of the sleeve for a fractional change of speed is the measure of sensitivity of a governor. A governor is also said to be sensitive if for a given displacement of the sleeve, the fractional change of speed is smaller. This definition of sensitiveness is quite satisfactory when the governor is considered as an independent mechanism. But when the governor is fitted to an engine the practical requirement is simply that the change of equilibrium speed from the full load to zero load position of the sleeve, should be as small a fraction as possible of the mean equilibrium speed. The actual displacement of the sleeve is immaterial, provided that it is sufficient to change the energy supplied to the engine by the required amount. For this reason sensitiveness is also *defined as the ratio of difference between the maximum and minimum equilibrium speeds to the mean equilibrium speed.*

Let N_1 = Minimum equilibrium speed corresponding to full-load condition,

N_2 = Maximum equilibrium speed corresponding to zero load condition

$$N = \text{Mean equilibrium speed} = \frac{N_1 + N_2}{2}$$

Then sensitiveness of the governor

$$= \frac{\text{Difference of maximum and minimum equilibrium speeds}}{\text{Mean equilibrium speed}}$$

$$= \frac{N_2 - N_1}{N} = \frac{N_2 - N_1}{\frac{N_1 + N_2}{2}} = \frac{2(N_2 - N_1)}{N_1 + N_2}.$$

15.9.2. Stability

A governor is said to be stable when for each speed there is only one radius of rotation of the governor balls at which the governor is in equilibrium. The speed should be within the working range of the governor.

15.9.3. Isochronism

A governor is said to be *isochronous* if the equilibrium speed is constant for all radii of rotation of the balls within the working range. This means that when radius of rotation changes from minimum radius to maximum radius, the equilibrium speed remains constant.

Let r_1 = Minimum radius of rotation

r_2 = Maximum radius of rotation

N_1 and N_2 = corresponding speeds.

Then for isochronism, $N_1 = N_2$.

15.9.4. Hunting

If the speed of the engine controlled by the governor fluctuates continuously above and below the mean speed, the governor is said to be hunting. This is caused by a too sensitive governor which changes the fuel supply by a large amount when a small change in speed of rotation takes place. For example, if a slight load increases on the engine, the speed of the engine will decrease. If the governor is very sensitive, the governor sleeve immediately falls to its lowest position. This will open the control valve wide and the supply of fuel to the engine will now be in excess of its requirements, so that the speed will rapidly increase again and the sleeve will rise to its highest position. Due to this movement of the sleeve, the control valve will cut off the fuel supply to the engine and thus the engine speed will begin to fall once again. This cycle is repeated indefinitely. Such a governor would admit either the maximum or minimum amount of fuel and could not possibly admit an amount of fuel between these two extremes. The effect of this will be to cause wide fluctuations in the engine speed or in other words, the engine will hunt.

15.10. Governor Effort and Power

Governor Effort. The effort of the governor is the force exerted by the governor at the sleeve and the sleeve tends to move. When the speed is constant, the force exerted to the sleeve is zero as the sleeve does not tend to move and hence at constant speed, the effort of the governor is zero. But when the speed changes, the sleeve tends to move to its new equilibrium position and hence a force is exerted on the sleeve. This force gradually diminishes to zero as the sleeve moves to the equilibrium position corresponding to new speed. *The mean force exerted on the sleeve during a given change of speed, is known as the effort of the governor.* The given change of speed is taken generally as 1%. Hence the effort is defined as the force exerted on the sleeve for 1% change of speed.

Governor Power. The power of a governor is defined as the work done at the sleeve for a given percentage change of speed. Hence the power of a governor is the product of the governor effort and the displacement of the sleeve. Mathematically,

Power of a governor = Governor effort × displacement of sleeve.

15.10.1. Method of Determining the Effort and Power of a Governor

The effort and power of a governor may be determined by the following method. Let us apply this method on Porter governor. The same principle will be used for any other type of governor.

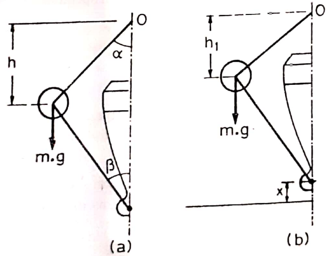


Fig. 15.19

Fig. 15.19 shows the two positions of a Porter governor.

Let N = Equilibrium speed corresponding to configuration shown in Fig. 15.19 (*a*)

W = Weight of sleeve = $M \times g$ where M is the mass of the sleeve

h = Height of governor corresponding to speed N

$c.N$ = Increase of speed

c = A factor which when multiplied to equilibrium speed, gives the increase in speed.